Hyperbolic type metrics and quasiconformal maps

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Abstract

- Review of some topics in geometric function theory connected with metrics in one way or other. The underlying space is a subset of Euclidean space $\mathbb{R}^n$, but some considerations make sense e.g. in Hilbert spaces, manifolds or even metric spaces.
- Some of our metrics are generalizations of the hyperbolic metric that have been studied in connection with qc mappings.
- We call these "hyperbolic type metric".

1.Introduction

- Based, in part, on 2013-2015 papers/preprints VW1, VW2, KVZ, CHKV, HVW, HVZ, VZ
- Why are metrics important? We can use metrics to gain better understanding of maps and "distortion". Various metrics such as chordal, Euclidean, hyperbolic, quasihyperbolic metrics are recurrent in geometric theory of functions.

- One of the main problems in the theory of $K-$qc maps in $\mathbb{R}^n$, $n \geq 2$, is what happens when $n = 2$ and $K \to 1$. Naturally, one expects to get results which are sharp or asymptotically sharp. For the $K-$qr/ $K-$qc versions of the Schwarz Lemma such results are known, if we use hyperbolic metric $\rho_{\mathbb{B}^n}/\rho_{\mathbb{H}^n}$ of $\mathbb{B}^n$/ $\mathbb{H}^n$. Here our goal is to explore whether and to what extent these results also hold for other metrics.
- Unlike the case $n = 2$, for $n \geq 3$ one cannot expect conformally invariant results. Therefore "quasi-invariance" is desiderable.

F.Klein's Erlangen Program 1872 for geometry

- Γ is the group of isometries
- use isometries ("rigid motions") to study geometry
- two configurations are considered equivalent if they can be mapped onto each other by an element of Γ
- the basic "models" of geometry are
  - (a) Euclidean geometry of $\mathbb{R}^n$
  - (b) hyperbolic geometry of the unit ball B^n in $\mathbb{R}^n$
  - (c) spherical geometry (Riemann sphere)

The main examples of Γ are subgroups of Möbius transformations of $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$.

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Example: Rigid motions and invariant metrics

X	Γ	metric
G	$\mathcal{M}(G)$	ρ_G hyperbolic metric, $G = B^n, \mathbb{H}^n$
$\overline{\mathbb{R}}^n$	$\mathrm{Iso}(\overline{\mathbb{R}}^n)$	q chordal metric
$\mathbb{R}^n$	transl.	$\lvert\cdot\rvert$Euclidean metric

Conformal invariance

Klein's program had enormous influence not only on geometry but also on function theory. Conformal invariance became a paradigma or a leading idea of geometric function theory.

- Invariant versions of Schwarz lemma
- Harmonic measure
- Extremal length of curve family, Ahlfors-Beurling

From Erlangen to Quasiworld

For the purpose of studying mappings defined in subdomains of $\mathbb{R}^n$, we must go beyond Erlangen, to the quasiworld, in order to get a rich class of mappings.

$$\begin{array}{rcl} \text{Conformal} & \longrightarrow & \text{"Quasiconformal"} \\ \text{Invariance} & \longrightarrow & \text{"Quasi-invariance"} \\ \text{Unit ball} & \longrightarrow & \text{"Classes of domains"} \\ \text{Smooth} & \longrightarrow & \text{"Nonsmooth"} \\ \text{Hyperbolic metric} & \longrightarrow & \text{Hyperbolic type metric"} \end{array}$$

Outline and future scenario

Outline

- Review various metrics such as hyperbolic metric, visual angle metric,
- Study how they behave under qc maps.

Scenario for further work

- Geometry of balls of small radii: convexity, smoothness of boundary, topological properties (Klén, Rasila, Talponen)
- Given two metrics, are Lipschitz (or uniformly continuous) maps qc and vice versa?
- Characterize those domains for which two given metrics are equivalent. (Well-know special case: uniform domains.)

2. Preliminary results

For $x \in \mathbb{R}^n$ and $r > 0$ let

$$\begin{aligned} B^n(x,r) &= \{ z \in \mathbb{R}^n : |x-z| < r \}, \\ S^{n-1}(x,r) &= \{ z \in \mathbb{R}^n : |x-z| = r \} \end{aligned}$$

denote the ball and sphere, respectively, centered at x with radius r. Also: $B^n(r) = B^n(0,r)$, $S^{n-1}(r) = S^{n-1}(0,r)$, $\mathbb{B}^2 = B^n(1)$, $S^{n-1} = S^{n-1}(1)$.

For distinct points $a, b, c, d \in \overline{\mathbb{R}^n}$ the absolute ratio is

$$|a,b,c,d| = \frac{q(a,c)q(b,d)}{q(a,b)q(c,d)} \left(= \frac{|a-c||b-d|}{|a-b||c-d|} \right).$$

It is invariant under Möbius transformations.

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Weighted metric.

Let $G \subset X$ be a domain and $w : G \to (0,\infty)$ continuous. For fixed $x, y \in G$, define

$$d_w(x,y) = \inf\{\ell_w(\gamma) : \gamma \in \Gamma_{xy},\, \ell(\gamma) < \infty\}, \ell_w(\gamma) = \int_\gamma w(\gamma(z))|dz|.$$

It is easy to see that d_w defines a metric on G and (G, d_w) is a metric space.

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Corner stones of distortion theory

Sharp K-qc Schwarz lemma and linear dilatation bound

- $f : \mathbb{B}^n \to f(\mathbb{B}^n) \subset \mathbb{B}^n, f(0) = 0$ K-qc $\Rightarrow$

$$|f(x)| \leq 2^{1-1/K} K|x|^{\alpha}, \alpha = K^{1/(1-n)}.$$

- $f : \mathbb{R}^n \to \mathbb{R}^n, f(0) = 0$ K-qc $K \in [1, 2] \Rightarrow$

$$fS^{n-1} \subset B^n(0, u) \setminus B^n(0, v) \& \frac{u}{v} \leq exp(30\sqrt{K-1}).$$

Below we will give for $n = 2$ a refined version of the Schwarz lemma.

Qspheres map into annuli

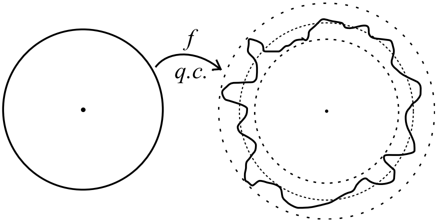

Figure : Spheres map into annuli under K-qc, $u/v \to 1, K \to 1$

This Euclidean distortion theory result has applications to Hausdorff dimension of quasispheres (Mattila-V 1990, Prause 2007, Badger-Gill-Rohde-Toro, 2014)

3. The hyperbolic metric

Four definitions of the hyperbolic metric ρ_{B^n}

1. Weighted metric: $\rho_{B^n} = m_w$, $w(x) = \frac{2}{1-|x|^2}$.
2. Explicite formula: $\sinh^2 \frac{\rho_{B^n}(x,y)}{2} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}$.
3. Absolute ratio:
   $\rho_{B^n}(x,y) = \sup\{\log|a,x,y,d| : a,d \in \partial B^n\}$.
4. Endpoints of geodesics: $\rho_{B^n}(x,y) = \log|x_*,x,y,y_*|$.
   Here x_*, y_* are the points of intersection of the circular arc perpendicular to ∂B^n, with ∂B^n.

All these four definitions are equivalent and have their counterparts for $\mathbb{H}^n$.

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Hyperbolic circles as Apollonian circles

Recall that $x^* = x/|x|^2$.

Apollonian circles and hyperbolic metric

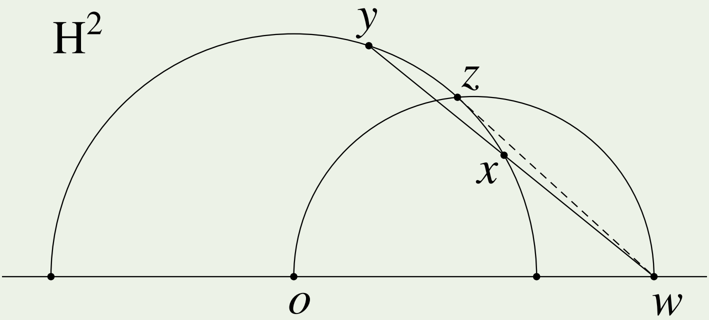
For $x \in \mathbb{B}^2 \setminus \{0\}$, the hyperbolic circle centered at x is an Apollonian circle with the base points x, x^*.

Hyperbolic geodesic

Hyperbolic geodesics are arcs of circles, which are orthogonal to the boundary of the domain. For any two distinct points $a, b \in \mathbb{B}^2$ the hyperbolic geodesic segment $J[a, b]$ is unique.

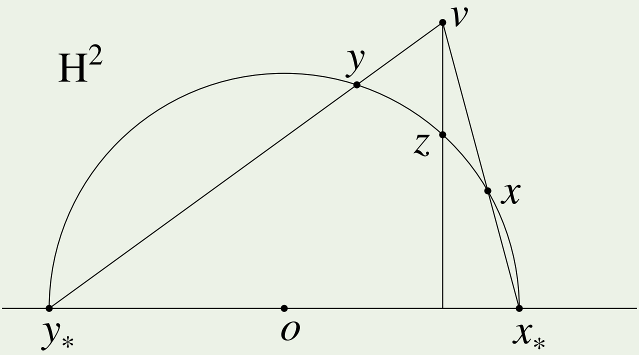
The constructions of midpoint z of the hyperbolic segment $J[x, y]$ in H^2, G. Wang [VW1]

Case 2. Method I.



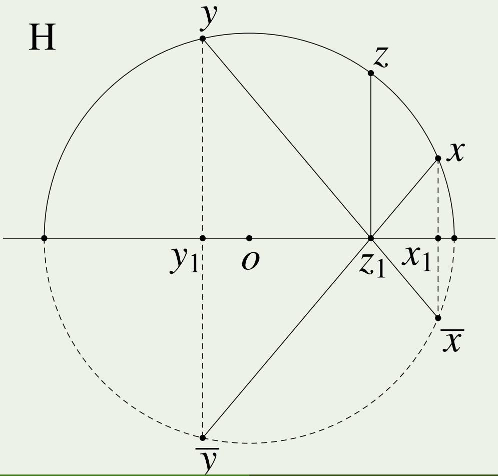
The constructions of midpoint z of the hyperbolic segment $J[x, y]$ in H^2

Case 2. Method II.



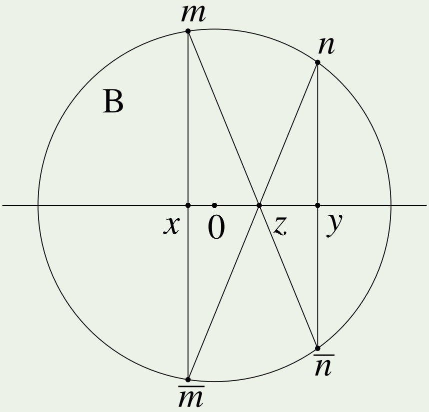
The constructions of midpoint z of the hyperbolic segment $J[x, y]$ in H^2

Case 2. Method IV.



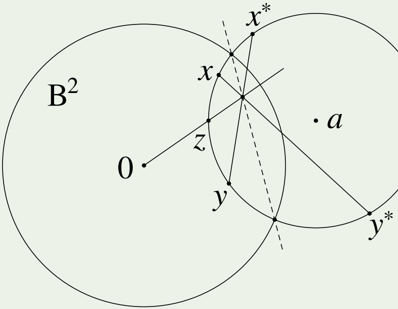
The constructions of midpoint z of the hyperbolic segment $J[x, y]$ in $\mathbb{B}^2$

Case 1.



The constructions of midpoint z of the hyperbolic segment $J[x, y]$ in $\mathbb{B}^2$

Case 2. Method II.



$$u = \frac{y(1-|x|^2)+x(1-|y|^2)}{1-|x|^2|y|^2}$$

Hyperbolic distances in $\mathbb{H}^2$ and $\mathbb{B}^2$

For points $e^{i\alpha}$, $e^{i\beta}$, $0 < \alpha < \beta < \pi$ we see by the definition of the absolute ratio that

$$|1, e^{i\alpha}, e^{i\beta}, -1|^2 = |1, \cos\alpha, \cos\beta, -1|.$$

This can be understood as an identity between the hyperbolic distances as follows

$$2\rho_{\mathbb{H}^2}(e^{i\alpha}, e^{i\beta}) = \rho_{\mathbb{B}^2}(\cos\alpha, \cos\beta)$$

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Points on $\partial\mathbb{B}^2$ and their projections on $(-1,1)$

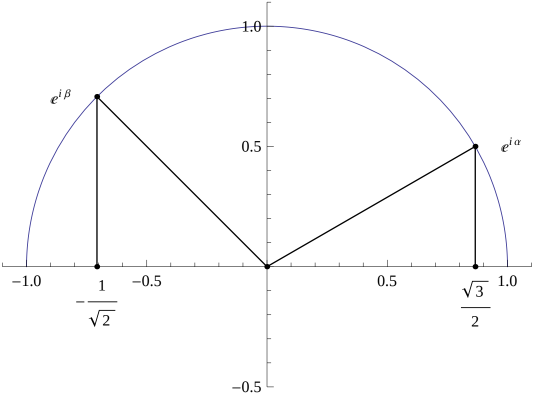

Figure : $|1, e^{i\alpha}, e^{i\beta}, -1|^2 = |1, \cos\alpha, \cos\beta, -1|.$

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Möbius map of $\mathbb{B}^2$ onto S^2_+

Connection between $\rho_{\mathbb{B}^2}$ and $\rho_{\mathbb{H}^3}$

The Möbius transformation

$$h(x) = -e_3 + \frac{2(x+e_3)}{|x+e_3|^2}, \quad x \in \mathbb{B}^2,$$

maps $\mathbb{B}^2$ onto S^2_+ in such a way that $h(\partial\mathbb{B}^2) = \partial\mathbb{B}^2$ and circular arcs of $\mathbb{B}^2$ perpendicular to $\partial\mathbb{B}^2$ are mapped onto circular arcs of $\mathbb{H}^3$ perpendicular to $\partial\mathbb{H}^3$. Thus we see that h provides a connection between hyperbolic geometries of $\mathbb{B}^2$ and $\mathbb{H}^3$.

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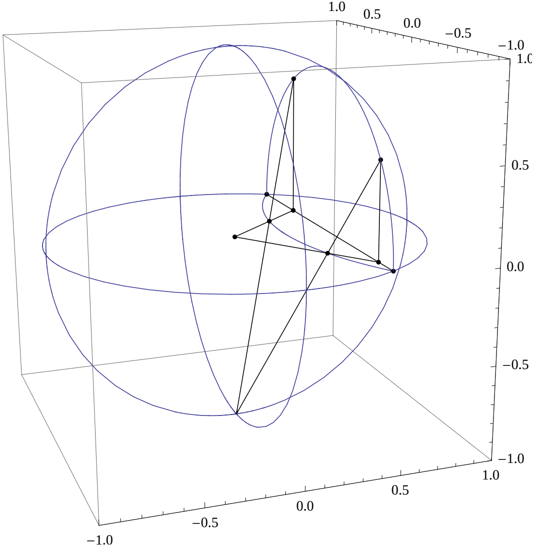
Hyperbolic geometries of $\mathbb{B}^2$ and $\mathbb{H}^2$



Question

Does the above picture have a counterpart when $\mathbb{B}^2$ is replaced by a convex domain (say an ellipse or a square) and the hyperbolic metric is replaced by the quasihyperbolic metric?

Corollary

Joining the end points of an orthogonal arc with a chord one can bisect hyperbolic distance.

$$2\rho_{\mathbb{B}^2}(0,c) = \rho_{\mathbb{B}^2}(0,c_2).$$

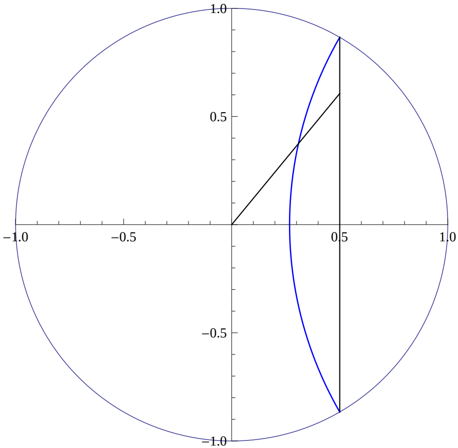


Figure : Orthogonal arc bisects the radial segment in hyperbolic geometry

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4. Apollonian and Möbius invariant metric

In this section we discuss briefly two metrics, the Apollonian metric α_G and a Möbius invariant metric δ_G introduced by P. Seittenranta. For the case of the unit ball, both coincide with the hyperbolic metric. For other domains they are quite different: while δ_G is always a metric, for domains with ∂G subset of $(n-1)$-dimensional plane, α_G may be a pseudometric. The Apollonian metric was introduced in 1934 by D. Barbilian, but forgotten for many years. A. Beardon rediscovered it independently in 1998 and thereafter it has been studied very intensively by many authors: see, e.g., Z. Ibragimov, P. Hästö , S. Ponnusamy , S. Sahoo . See also D. Herron, W. Ma and D. Minda.

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Apollonian metric of $G \subsetneq \mathbb{R}^n$

$$\alpha_G(x,y) = \sup\{\log|a,x,y,b| : a,b \in \partial G\}.$$

- α_G agrees with ρ_G, if G equals B^n or H^n.
- $\alpha_{hG}(hx,hy) = \alpha_G(x,y)$ for $h \in \mathcal{GM}(\mathbb{R}^n)$
- α_G is a pseudometric if ∂G is "degenerate"

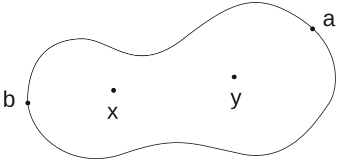

Figure : Apollonian metric

Seittenranta's metric δ_G

For $x, y \in G \subsetneq \mathbb{R}^n$, Seittenranta's metric (PhD thesis 1997) is defined by

$$\delta_G(x,y) = \sup_{a,b \in \partial G} \log\{1 + |a,x,b,y|\}\,.$$

Facts

1. The function δ_G is a metric.
2. δ_G agrees with ρ_G, if G equals B^n or H^n.
3. It follows from the definitions that $\delta_{\mathbb{R}^n \setminus \{a\}} = j_{\mathbb{R}^n \setminus \{a\}}$ for all $a \in \mathbb{R}^n$.
4. $\alpha_G \le \delta_G \le \log(e^{\alpha_G} + 2) \le \alpha_G + 3$. The first two inequalities are best possible for δ_G in terms of α_G only.

5. Quasihyperbolic metric

Let G be a proper subdomain of $\mathbb{R}^n$. For all $x, y \in G$, the quasihyperbolic metric k_G is defined as

$$k_G(x,y) = \inf_{\gamma} \int_{\gamma} \frac{1}{d(z,\partial G)} |dz|,$$

where the infimum is taken over all rectifiable arcs γ joining x to y in G (Gehring-Palka 1976).

Distance ratio metric.

For a proper open subset $G \subset \mathbb{R}^n$ and for all $x, y \in G$, the distance ratio metric j_G is defined as

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x,\partial G), d(y,\partial G)\}}\right).$$

We also write $d(x) = d(x,\partial G)$.

Uniform domains

We always have for all $x, y \in G$

$$k_G(x,y) \geq j_G(x,y)\,.$$

The opposite inequality defines uniform domains.

Def.

A domain G in $\mathbb{R}^n$ is a uniform domain, if there exists $C \geq 1$ such that for all $x, y \in G$

$$k_G(x,y) \leq C\, j_G(x,y)\,.$$

Idea: Generalized uniform domain. Given a pair of metrics d_1, d_2 on a domain G we can ask under which conditions there exists a constant $C \geq 1$ such that for all $x, y \in G$

$$1/C \leq d_1(x,y)/d_2(x,y) \leq C\,.$$

Quasiconformal maps

Theorem, Gehring-Osgood, 1979

A qc homeomorphism $f : G \to G' = fG$ satisfies for all $x, y \in G$

$$k_{G'}(f(x), f(y)) \le C \max\{k_G(x,y)^{1/C}, k_G(x,y)\}$$

where $C \ge 1$ is a constant.

Theorem, Väisälä, 1990's

A homeomorphism $f : D \to D' = fD$ for which there exists a constant $C \ge 1$ such that for all subdomains $G \subset D$ and for all $x, y \in G$

$$k_{G'}(f(x), f(y)) \le C \max\{k_G(x,y)^{1/C}, k_G(x,y)\}$$

holds, is quasiconformal.

6. Modulus of curve family and metric

The curve family joining two sets E, F in G is denoted by $\Delta(E, F; G)$ and its modulus by $M\big(\Delta(E, F; G)\big)$. The modulus is conformal invariant. A homeo $f : G \to G'$ is K-qc if

$$M(\Gamma)/K \le M(f\Gamma) \le KM(\Gamma), \quad \forall \Gamma \subset G\,.$$

In this section we shall introduce two other conformal invariants, the modulus metric $\mu_G(x, y)$ and its "dual" quantity $\lambda_G(x, y)$, where G is a domain in $\mathbb{R}^n$ and $x, y \in G$. Both μ_G and $\lambda_G(x, y)$ are functionally related to the hyperbolic metric ρ_G if $G = \mathbb{B}^2$, while for a general domain μ_G reflects the “capacitary geometry” of ∂G in a delicate fashion.

Conformal invariant λ_G Ferrand 1973

If G is a proper subdomain of $\mathbb{R}^n$, then for $x, y \in G$ with $x \neq y$ we define

$$\lambda_G(x,y) = \inf_{C_x, C_y} M\Big(\Delta(C_x, C_y; G)\Big)$$

where $C_z = \gamma_z[0,1)$ and $\gamma_z\colon [0,1) \to G$ is a curve such that $\gamma_z(0) = z$ and $\gamma_z(t) \to \partial G$ when $t \to 1$, $z = x, y$. It follows from conformal invariance of the modulus that λ_G is invariant under conformal mappings of G. That is, $\lambda_{fG}(f(x), f(y)) = \lambda_G(x,y)$, if $f\colon G \to fG$ is conformal and $x, y \in G$ are distinct.

Conformal invariant μ_G

If G is a proper subdomain of $\mathbb{R}^n$, then for $x, y \in G$ with $x \neq y$ we define

$$\mu_G(x, y) = \inf_{C_{x,y}} M\big(\Delta(C_{x,y}, \partial G; G)\big)$$

where $C_{x,y}$ is a continuum joining x and y. It follows from conformal invariance of the modulus that μ_G is invariant under conformal mappings of G. That is, $\mu_{fG}(f(x), f(y)) = \mu_G(x, y)$, if $f : G \to fG$ is conformal and $x, y \in G$ are distinct.

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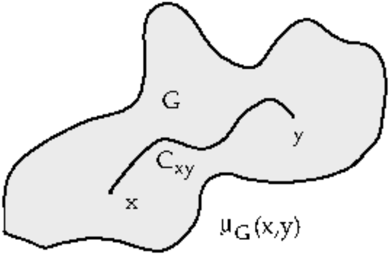

Figure : The conformal invariants λ_G and μ_G .

$$\lambda_G(x,y) = \inf_{C_x, C_y} M\big(\Delta(C_x, C_y; G)\big)$$

$$\mu_G(x,y) = \inf_{C_{x,y}} M\big(\Delta(C_{x,y}, \partial G; G)\big)$$

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Remark

- J. Ferrand proved that $\lambda_G(x,y)^{1/(1-n)}$ is a metric thus answering a question in [Vu1].
- It is easy to see that $\mu_G(x,y)$ is either a metric or identically 0.
- $\lambda_G(x,y)^{1/(1-n)}$, $G = \mathbb{R}^n \setminus \{0\}$, reduces to Teichmüller's problem and $\mu_G(x,y)$, $G = \mathbb{B}^n$, to Grötzsch's problem.

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Ferrand's problem

It follows from the definition of a qc map that such maps are bilipschitz in both the ${\lambda_G}^{1/(1-n)}$ and μ_G metrics. Let $f : G \to fG$ be a homeo.

Ferrand's problem, 1973

Does $\lambda_{fG}(f(x), f(y))^{1/(1-n)} \le C\lambda_G(x, y)^{1/(1-n)}$ for all $x, y \in G$, imply that f is qc?

Answer by Ferrand-Martin-Vuorinen 1996: No. Yes, if we require the same condition also for all subdomains.

Klén-Vuorinen-Zhang 2015

λ_G-isometries are qc. If $G = \mathbb{R}^n \setminus \{0\}$, then λ_G-isometries are Möbius.

7. Triangular ratio metric [CHKV],[HVZ]

The triangular ratio metric is defined as follows for a domain $G \subset \mathbb{R}^n$ and $x, y \in G$:

$$s_G(x,y) = \sup_{z \in \partial G} \frac{|x-y|}{|x-z|+|z-y|} \in [0,1].$$

Theorem

For $x, y \in \mathbb{B}^n$ we have

$$\tanh\left(\frac{\rho_{\mathbb{B}^n}(x,y)}{4}\right) \le s_{\mathbb{B}^n}(x,y) \le \tanh\left(\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\right).$$

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Corollary

- If $f : \mathbb{H}^n \to \mathbb{H}^n$ is a Möbius transformation onto $\mathbb{H}^n$, then for all $x, y \in \mathbb{H}^n$,

$$s_{\mathbb{H}^n}(f(x), f(y)) = s_{\mathbb{H}^n}(x, y).$$

- If $f : \mathbb{B}^n \to \mathbb{B}^n$ is a Möbius transformation onto $\mathbb{B}^n$, then for all $x, y \in \mathbb{B}^n$,

$$s_{\mathbb{B}^n}(f(x), f(y)) \le 2 s_{\mathbb{B}^n}(x, y).$$

Open problem

If $f : \mathbb{B}^n \to \mathbb{B}^n$ is a Möbius transformation onto $\mathbb{B}^n$, is it true that for all $x, y \in \mathbb{B}^n$,

$$s_{\mathbb{B}^n}(f(x), f(y)) \leq (1 + |f(0)|)s_{\mathbb{B}^n}(x, y).$$

8. Visual angle metric [KLVW], [VW1],[VW2], [HVW]

Definition

For a domain $G \subsetneq \mathbb{R}^n$, $n \geq 2$, and $x, y \in G$ the visual angle metric is defined by

$$v_G(x,y) = \sup\{\measuredangle(x,z,y) : z \in \partial G\} \in [0,\pi].$$

∂G is not a proper subset of a line.

This metric was introduced and studied in [KLVW], in the PhD thesis of G. Wang.

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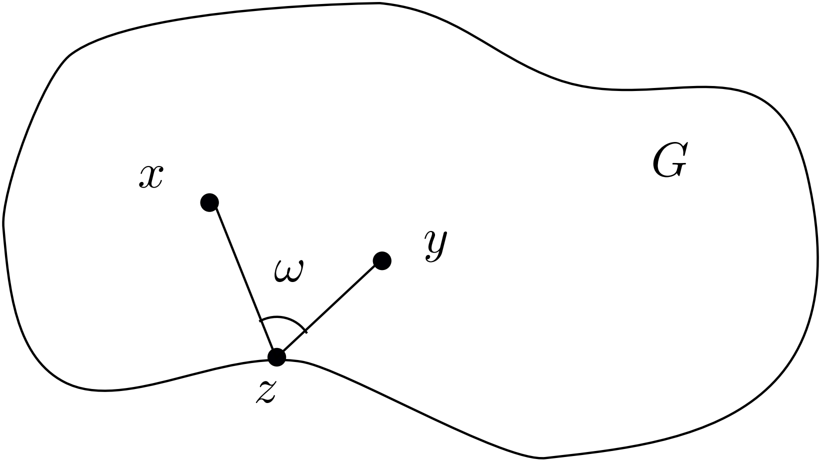

Figure : $v_G(x,y) = \sup\{\measuredangle(x,z,y) : z \in \partial G\}$

Theorem [Bhayo-V], [VW2]

If $f : \mathbb{B}^2 \to \mathbb{R}^2$ is a K-qc map with $f\mathbb{B}^2 \subset \mathbb{B}^2$ and ρ is the hyperbolic metric of $\mathbb{B}^2$, then

$$\rho_{\mathbb{B}^2}(f(x), f(y)) \le c(K) \max\{\rho_{\mathbb{B}^2}(x,y),\, \rho_{\mathbb{B}^2}(x,y)^{1/K}\}$$

for all $x, y \in \mathbb{B}^2$, where $c(K) = 2\,\mathrm{arth}(\varphi_K(\mathrm{th}\frac{1}{2}))$ and, in particular, $C(1) = 1$.

Note that here 2 is best possible.
Agard and Gehring have studied also change of angles under qc maps.

Theorem, [VW2]

If $f:\mathbb{B}^2\to\mathbb{R}^2$ is a K-qc map with $f\mathbb{B}^2\subset\mathbb{B}^2$, then

$$v_{\mathbb{B}^2}(f(x),f(y))\le C(K)\max\{v_{\mathbb{B}^2}(x,y),\, v_{\mathbb{B}^2}(x,y)^{1/K}\}$$

for all x, $y\in\mathbb{B}^2$, where $C(K)=2\cdot 4^{1-1/K}$ and $C(1)=2$.

Agard and Gehring have studied also change of angles under qc maps.

9. Application of metrics

Teichmüller's (1913- 1943) problem

Let G be a proper subdomain of $\mathbb{R}^n$ ($n \geq 2$), and let

$$\mathrm{Id}_K(\partial G) = \{f : \mathbb{R}^n \to \mathbb{R}^n \text{ is } K-\text{quasiconformal} :$$

$$f(x) = x, \forall x \in \mathbb{R}^n \setminus G\}.$$

Teichmüller, 1944

For $x \in D, f \in Id_K(\partial D)$, we have

$$\log K(f) \geq h_D(x, f(x))$$

where h_D is the hyperbolic metric of $D = \mathbb{R}^2 \setminus \{0, 1\}$.

Note. This result does not tell how to estimate $h_D(x, f(x))$.
Bonfert-Taylor, Canary, Taylor, Bridgeman Riemann surfaces

Vuorinen-Zhang 2014

Convex domains

Let $D \subsetneq \mathbb{R}^n$ be a convex domain and $f \in \mathrm{Id}_K(\partial D)$, $K \in [1, K_n)$. Then, for all $x \in D$,

$$\log\left(1 + \frac{|x - f(x)|}{\min\{d(x), d(f(x))\}}\right) = j_D(x, f(x)) \le 4\sqrt{K-1}\,.$$

Additional results: [Manojlovic-V] 2011, [Bhayo-V] 2011, [Li-V-X.Wang] (Banach spaces) 2014, Prause 2014, $n = 2$.

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Thank you!