Analysis of the PML equations in general convex geometry

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Abstract

In this work, we study a mesh termination scheme in acoustic scattering, known as the PML method for Perfectly Matched Layer. The main result of the paper is the following: Assume that the scatterer is contained in a bounded and strictly convex artificial domain. We surround this domain by a perfectly matched layer of constant thickness. On the peripheral boundary of this layer, a homogenous Dirichlet condition is imposed. We show in this paper that the resulting boundary value problem for the scattered field is uniquely solvable for all wave numbers and the solution witihin the artificial domain converges exponentially fast toward the full space scattering solution when the layer thickness is increased. The proof is based on the idea of interpreting the PML medium as a complex stretching of the coordinates in $\mathbb{R}^n$ and on the use of complexified layer potential techniques.

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1 Introduction

In computational electromagnetism and acoustics, the finite element or finite difference mesh termination in scattering problems without generating exessive reflection error is a widely studied problem. One possible approach is to surround the scatterer by a non-reflecting fictitious material layer that absorbs quickly the scattered waves. This approach, known as PML for Perfectly Matched Layer, was suggested in the works of Bérenger (see [2]). While the PML approach has been the subject of numerous engineering papers, only little theoretical analysis has been done on it. One central question is the solvability of the PML equations in a truncated region as well as the convergence of the PML solutions towards the true scattering solution as the computational region increases. These questions have been studied in [5] and [13]. Another important problem is the optimization of the fictitious material parameters in order to get highest possible numerical fidelity of the solution. This latter question is the subject of the article [6].

The present work is a contribution to the theoretical analysis of the existence and convergence of the PML solutions. We extend and continue here the work started in [13]. In the cited work, it was shown that with a very special choice of the fictitious absorbing coefficient, the PML equation is solvable in a circular domain of the plane and that the solution converge exponentially towards the scattering solution near the scatterer as the mesh termination surface is pushed towards infinity. In the present work, we first of all relax the geometry so that the computational region is a convex domain surrounding the scatterer. What is more, we are able to get rid of the very restrictive choice of the absorbing coefficient used in [13]. In the light of the optimization results of [6], this relaxation seems to be quite important.

Methodologically, the starting point in this work is the complex stretching of the spatial coordinates (see [3], [4], [5] and [13]). A large part of this work is dedicated to the derivation of what could be called *complexified scattering theory* which may have interest in its own right. We refer here to article [15] where similar ideas surface in a different context.

2 Main results and an outline

We consider the scattering problem of the Helmholtz equation in $\mathbb{R}^n$. For the sake of definiteness, the discussion is restricted to the scattering by a sound-hard obstacle, although the results apply to general scattering problems by a bounded scatterer.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain representing the scatterer, and assume that Ω has a connected complement. The scattering problem is to find a solution u of the system

$$
\begin{aligned}
(\Delta + k^2)u &= 0 \text{ in } \mathbb{R}^n \setminus \overline{\Omega}, && (1)\\
\frac{\partial u}{\partial n}\Big|_{\partial\Omega} &= g \in H^{-1/2}(\partial\Omega), && (2)\\
\lim_{r\to\infty} r^{(n-1)/2}\left(\frac{\partial u}{\partial r} - iku\right) &= 0 \text{ uniformly in } \hat{x}, && (3)
\end{aligned}
$$

where $r = |x|$, and $\hat{x} = x/r$, $r \neq 0$. For the radiation condition (3) to make sense, it is undersood that the function u is continuously differentiable outside some ball. It is well-known (see e.g. [7], [8], [14], [16]), that the problem (1)–(3) has a unique (weak) solution $u \in H^1_{\text{rad}}(\mathbb{R}^n \setminus \overline{\Omega})$, where

$$
H^1_{\text{rad}}(\mathbb{R}^n \setminus \overline{\Omega}) = \{u \mid u \in H^1(B_R \setminus \overline{\Omega}) \text{ for all } R > 0,\, u \text{ satisfies (3)}\},
$$

where B_R denotes a ball in $\mathbb{R}^n$ of radius R and center at the origin. We refer to this unique solution as the *scattering solution* and denote it by u_{sc}.

Assume that one seeks to solve the problem (1)–(3) numerically e.g. by FEM in the vicinity of the scatterer. Denote by $D \in \mathbb{R}^n$ a strictly convex domain with a C^2–boundary such that $\overline{\Omega} \subset D$. Here, $D \setminus \overline{\Omega}$ is the domain where the approximate solution is requested. The *mesh termination problem* is to impose a proper boundary condition on the artificial boundary ∂D such that possible reflections by this boundary do not contaminate too hevily the approximate scattering solution in $D \setminus \overline{\Omega}$. The literature concerning this problem is vast and therefore we do not try to cover it here, but mention only the relatively recent articles [9] and [11] as examples of possible approaches different from considered here.

The idea of the Perfectly Matched Layer approach is to surround the computational domain D by a fictitious layer that has minimal reflection and strong absorption properties, and extend the FEM computation from D to this buffering layer. The obvious advantage of this approach, compared to e.g. the use of absorbing boundary conditions on ∂D directly, is that one can use simple boundary conditions (e.g. Dirichlet or Neumann) at the peripheral boundary of the perfectly matched layer and so the implementation does not differ essentially from the implementation of a standard interior problem.

In this work, we follow the ideas of [3], [4] and [5] and define the PML layer through a complex stretching of the exterior domain $\mathbb{R}^n \setminus \overline{D}$.

Let $x \in \mathbb{R}^n \setminus D$. We define $h(x) \geq 0$ and $p(x) \in \partial D$ by

$$h(x) = \operatorname{dist}(x, \partial D) = |x - p(x)|.$$

If $n(x)$ denotes the exterior unit normal vector of ∂D at $p(x)$, we may represent x in a unique way as

$$x = p(x) + h(x)n(x). \tag{4}$$

Further, let $\tau : [0, \infty) \to [0, \infty)$ be a twice continuously differentiable function with strictly increasing derivate τ' satisfying

$$\lim_{s\to\infty} \tau'(s) = \infty, \quad \tau(0) = \tau'(0+) = \tau''(0+) = 0. \tag{5}$$

Moreover, we assume that growth of τ' and τ'' is moderate in the sense that

$$\lim_{s\to\infty} e^{-\varepsilon\tau(s)}\tau'(s) = \lim_{s\to\infty} e^{-\varepsilon\tau(s)}\tau''(s) = 0. \tag{6}$$

for all $\varepsilon > 0$.

We start with the following definition, where the notation $\overline{\mathbb{C}}^{++} = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0,\ \operatorname{Im} z \geq 0\}$ is used.

Definition 2.1 *Let $s \in \overline{\mathbb{C}}^{++}$. We define a function*

$$a : \mathbb{R}^n \to \mathbb{R}^n, \quad a(x) = \begin{cases} 0, & x \in D \\ \tau(h(x))n(x)\ , & x \in \mathbb{R}^n \setminus \overline{D}, \end{cases}$$

and a mapping $F_s : \mathbb{R}^n \to \mathbb{C}^n$ by setting

$$F_s(x) = x + sa(x),\ x \in \mathbb{R}^n$$

Particularly, when $s \geq 0$ is real, F is C^2-diffeomorphism in $\mathbb{R}^n$.

The stretched $\mathbb{R}^n$ with parameter s is the submanifold of $\mathbb{C}^n$ given as

$$\Gamma_s = \{z \in \mathbb{C}^n \mid z = x + sa(x),\ x \in \mathbb{R}^n\}.$$

If $s = i$, we denote simply $\Gamma_i = \Gamma$.

We refer to the function F_s above as the *stretching function.*

The first step is to extend the scattering solution analytically to the stretched exterior domain.

Let Φ denote the fundamental solution of the Helmholtz equation in $\mathbb{R}^n$ satisfying the Sommerfeld radiation condition at infinity,

$$\Phi(x, y) = \frac{i}{4}\left(\frac{k}{2\pi|x-y|}\right)^{(n-2)/2} H^{(1)}_{(n-2)/2}(k|x-y|).$$

Using this fundamental solution, we define the radiative single and double layer potential operators as

$$S_{\partial\Omega,X}\varphi(x) = \int_{\partial\Omega} \Phi(x,y)\varphi(y)dS(y), \quad x \in X$$

and

$$K_{\partial\Omega,X}\psi(x) = \int_{\partial\Omega} \frac{\partial\Phi}{\partial n(y)}(x,y)\psi(y)dS(y), \quad x \in X,$$

where $X \subset \mathbb{R}^n$, $X \cap \partial\Omega = \emptyset$.

Since $u_{\mathrm{sc}} \in H^1_{\mathrm{rad}}(\mathbb{R}^n \setminus \overline{\Omega})$ satisfies the Helmholtz equation in the exterior domain $\mathbb{R}^n \setminus \overline{\Omega}$, u_{sc} can be represented in terms of the single– and double–layer potentials,

$$u_{\mathrm{sc}} = S_{\partial\Omega,\mathbb{R}^n\setminus\overline{\Omega}}\varphi + K_{\partial\Omega,\mathbb{R}^n\setminus\overline{\Omega}}\psi$$

with some densities φ and ψ.

Our aim is to analytically continue the potential operators to a neighborhood of Γ_s. Consider first the mapping

$$\rho(x) = |x| = (x^2)^{1/2} = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}, \quad x \in \mathbb{R}^n.$$

This function allows an analytic extension to $G \subset \mathbb{C}^n$,

$$G = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid z^2 = \sum_{j=1}^{n} z_j^2 \in \mathbb{C} \setminus (-\infty, 0)\}, \tag{7}$$

i.e., z^2 belonging to the complex plane with the branch cut along the negative real axis. This extension is denoted by the same symbol, $\rho : G \to \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. Since the Hankel function $H^{(1)}_{(n-2)/2}$ is analytic in $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$, we may define

$$\Phi(z, \zeta) = \frac{i}{4} \left(\frac{k}{2\pi \rho(z - \zeta)} \right)^{(n-2)/2} H^{(1)}_{(n-2)/2}(k\rho(z - \zeta)) \text{ for } z - \zeta \in G. \tag{8}$$

The analytic extension of the scattering solution is based on the following result.

Lemma 2.1 *The manifold $\Gamma_s \setminus \overline{\Omega} \in \mathbb{C}^n$ has a neighborhood $U_s \subset \mathbb{C}^n$ such that for all $y \in \partial\Omega$ and $z \in U_s$ $z - y \in G$.*

The proof of this lemma as well as other technical details of this section are postponed to later sections. We define now the analytic extensions of the layer potential operators $S_{\partial\Omega, U_s}$ and $K_{\partial\Omega, U_s}$ as

$$S_{\partial\Omega,U_s}\varphi(z) = \int_{\partial\Omega} \Phi(z, y)\varphi(y)dS(y), \quad z \in U_s, \tag{9}$$

$$\tag{10}$$

$$K_{\partial\Omega,U_s}\psi(z) = \int_{\partial\Omega} \frac{\partial\Phi}{\partial n(y)}(z, y)\psi(y)dS(y), \quad z \in U_s. \tag{11}$$

and further,

$$u(z, s) = S_{\partial\Omega,U_s}\varphi(z) + K_{\partial\Omega,U_s}\psi(z).$$

The properties of $u(z, s)$ are listed in the next lemma.

Lemma 2.2 *For the function $u(z, s)$, the following hold:*

(i) The function $z \mapsto u(z, s)$ is $\mathbb{C}^n$-analytic in U_s.

(ii) $u(\,\cdot\,, s)|_{D\setminus\overline{\Omega}} = u_{\mathrm{sc}}|_{D\setminus\overline{\Omega}}$.

(iii) The function $z \mapsto u(z,s)$ satisfies the complexified Helmholtz equation in U_s,

$$(\Delta_z + k^2)u(z,s) = 0,$$

where

$$\Delta_z = \partial_{z_1}^2 + \ldots + \partial_{z_n}^2.$$

Now we are ready to give the following definition where we set $s = i$ and denote simply $u(z,i) = u(z)$.

Definition 2.2 *Let u be an analytic function in a $\mathbb{C}^n$–neighborgood of Γ. We say that u satisfies* the Bérenger equation *corresponding to the exterior Helmholtz equation if*

$$[(\Delta_z + k^2)u(z)]|_{\Gamma\setminus\overline{\Omega}} = 0. \tag{12}$$

The function $u|_{\Gamma\setminus\overline{\Omega}}$ with the Neumann boundary condition (2) on $\partial\Omega$ is called the Bérenger solution *corresponding to the scattering problem with the Neumann boundary condition.*

The next step is to write the Bérenger equation explicitly in terms of the coordinates in $\mathbb{R}^n$ parametrizing the manifold Γ.

Theorem 2.1 *Let $s \in \overline{\mathbb{C}}^{++}$ and u be an analytic function defined in a neighborhood of $\Gamma_s \subset \mathbb{C}^n$. Then for $z \in \Gamma_s$*

$$\Delta_z u(z) = (\mathrm{div} H_s^{\mathrm{T}} H_s \mathrm{grad} - m_s^{\mathrm{T}} H_s \mathrm{grad})[u \circ F_s](F_s^{-1}(z)),$$

where

$$H_s = (I + s(Da)^{\mathrm{T}})^{-1}, \quad (m_s)_j = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}(H_s)_{j,k}.$$

Especially, the Bérenger equation $[(\Delta_z + k^2)u]|_\Gamma = 0$ assumes in $\mathbb{R}^n$ the form

$$(\mathrm{div} H^{\mathrm{T}} H \mathrm{grad} - m^{\mathrm{T}} H \mathrm{grad} + k^2)[u \circ F] = 0, \tag{13}$$

where $F = F_i$, $H = H_i$ and $m = m_i$.

In the sequel, we use the abbreviated notation

$$\tilde{\Delta}_s = \mathrm{div} H_s^{\mathrm{T}} H_s \mathrm{grad} - m_s^{\mathrm{T}} H_s \mathrm{grad}, \quad s \in \overline{\mathbb{C}}^{++}. \tag{14}$$

When $s = i$ we denote $\tilde{\Delta} = \tilde{\Delta}_s$

We point out that when $s \geq 0$ is real we have for $\phi \in C_0^\infty(\mathbb{R}^n)$

$$(\tilde{\Delta}_s + k^2)\phi(x) = (\Delta + k^2)[\phi \circ F_s^{-1}](F_s(x)). \tag{15}$$

Due to the asymptotic properties of the Hankel function $H^{(1)}_{(n-2)/2}$, the function $u|_\Gamma$ decays exponentially as $|x| \to \infty$. More precisely, we consider solutions that satisfy

$$\lim_{h(x)\to\infty} e^{\delta\tau(h(x))}|u(x)| = \lim_{h(x)\to\infty} e^{\delta\tau(h(x))}|\mathrm{grad}\, u(x)| = 0 \text{ uniformly in } \hat{x} \tag{16}$$

for some $\delta > 0$. It is natural to define a complexified analog of the space $H^1_{\mathrm{rad}}(\mathbb{R}^n \setminus \overline{\Omega})$,

$$H^1_{(\delta)}(\mathbb{R}^n \setminus \overline{\Omega}) = \{u \in H^1(\mathbb{R}^n \setminus \overline{\Omega}) \mid \text{ condition (16) holds}\},$$

where it is understood that near the infinity, u is continuously differentiable in order that the limits make sense. It is natural to consider the following *full space Bérenger problem* of finding an $u \in H^1_{(\delta)}(\mathbb{R}^n \setminus \overline{\Omega})$ such that

$$(\tilde{\Delta} + k^2)u = 0 \text{ in } \mathbb{R}^n \setminus \overline{\Omega}, \tag{17}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = g \in H^{-1/2}(\partial\Omega). \tag{18}$$

We have the following counterpart of the uniqueness of the scattering solution:

Theorem 2.2 *The problem (17)–(18) has a unique solution in $H^1_{(k-\varepsilon)}(\mathbb{R}^n \setminus \overline{\Omega})$, where $\varepsilon > 0$ is arbitrary. If u_{B} is this solution, we have $u_{\mathrm{B}}|_{D\setminus\overline{\Omega}} = u_{\mathrm{sc}}|_{D\setminus\overline{\Omega}}$.*

Observe that the analyticity assumption is no longer needed here. We call the unique solution of (17)–(18) in $H^1_{(k-\varepsilon)}(\mathbb{R}^n \setminus \overline{\Omega})$ the *full space Bérenger solution.* Thus, instead of solving numerically the original scattering problem, near the scatterer one can equivalently try to approximate Bérenger solution u_{B}. The main goal of this paper is to show that the truncation of the complexified problem to a bounded domain has an exponentially small effect on the solution near the scatterer. More precisely, let us denote

$$D(\rho) = D \cup \{x \in \mathbb{R}^n \setminus \overline{D} \mid h(x) < \rho\}, \quad \rho > 0,$$

i.e., $D(\rho)$ is obtained by adding a layer of thickness ρ around D. We define the *truncated Bérenger problem* as follows: Find a solution $u \in H^1(D(\rho) \setminus \overline{\Omega})$ satisfying

$$\begin{aligned}
(\tilde{\Delta} + k^2)u &= 0 \text{ in } D(\rho) \setminus \overline{\Omega}, && (19)\\
\frac{\partial u}{\partial n}\Big|_{\partial\Omega} &= g \in H^{-1/2}(\partial\Omega), && (20)\\
u|_{\partial D(\rho)} &= 0. && (21)
\end{aligned}$$

In practice, the system (19)–(21) is the one that is used for numerical approximation of the full space Bérenger solution u_{B} and thus the scattering solution u_{sc} in $D \setminus \overline{\Omega}$. The main result of this paper is the following.

Theorem A: *For any wavenumber $k > 0$ there exists a positive constant $\rho_0(k)$ such that for all $\rho \geq \rho_0(k)$, the truncated Bérenger problem (19)–(21) has a unique solution $\tilde{u}_{\mathrm{B}} = \tilde{u}_{\mathrm{B}}(\rho) \in H^1(D(\rho) \setminus \overline{\Omega})$. Moreover, this solution has the exponential approximation property*

$$\lim_{\rho \to \infty} e^{(k-\varepsilon)\tau(\rho)} \|u_{\mathrm{sc}} - \tilde{u}_{\mathrm{B}}(\rho)\|_{H^1(D\setminus\overline{\Omega})} = 0$$

for all $\varepsilon > 0$.

This result is an extension of the one obtained in [13], where D was a disc in $\mathbb{R}^2$ and the form of the stretching function was strongly limited.

We give here an outline of the proof of the Theorem A. Detailed discussion is postponed to the following sections.

The starting point is a near field radiation condition for the original scattering problem. Let $0 < \rho_1 < \rho_2$ and denote $D_j = D(\rho_j)$, $j = 1, 2$. Consider the problem of finding u satisfying

$$
\begin{aligned}
(\Delta + k^2)u &= 0 \text{ in } D_2 \setminus \overline{\Omega}, && (22)\\
\frac{\partial u}{\partial n}\Big|_{\partial\Omega} &= g \in H^{-1/2}(\partial\Omega), && (23)\\
u|_{\partial D_2} &= P(u|_{\partial D_1}), && (24)
\end{aligned}
$$

where the *double surface operator* $P : H^{1/2}(\partial D_1) \to H^{1/2}(\partial D_2)$ is defined with the layer potential operators as

$$
P = K_{\partial D_1, \partial D_2}(\frac{1}{2} + K_{\partial D_1})^{-1}. \tag{25}
$$

Here as throughout the rest of the paper we use the convention that layer potentials with only one subindex denote the weakly singular operators, i.e., here,

$$
K_{\partial D_1}\varphi(x) = \text{p.v.}\int_{\partial D_1} \frac{\partial \Phi}{\partial n(y)}(x,y)\varphi(y)dS(y), \quad x \in \partial D_1.
$$

The problem (22)–(24), where the usual Sommerfeld radiation condition is replaced by a near field condition, was introduced in the work [12], where the following result was proved:

Proposition 2.1 *Assume that ρ_1 and ρ_2 are so chosen that k^2 is not the Dirichlet eigenvalue of $-\Delta$ in $D_2 \setminus \overline{D}_1$. Then the problem (22)–(24) has a unique solution, and this solution coincides with the scatterind solution $u_{\rm sc}$ in D_2.*

The proof is a relatively simple application of layer potential techniques and will not be repeated here.

The first step towards the proof of Theorem A is to find a double surface operator analogous to P above for the full space Bérenger problem. From the definition (25), it is natural to seek a complexified analogue for the layer potential operators of the Helmholtz equation. Such an extension is discussed in detail in Section 5. An outline of the results is given below.

The following theorem defines the fundamental solution of the Bérenger operator and its formal transpose.

Theorem 2.3 *For $s \in \overline{\mathbb{C}}^{++}$, $x, y \in \mathbb{R}^n \setminus \Omega$, we have $F_s(x) - F_s(y) \in G$, the set defined in (7). Define the functions*

$$\begin{aligned} \tilde{\Phi}_s(x,y) &= J_s(y)\Phi(F_s(x), F_s(y)), \\ \tilde{\Phi}_s^{\mathrm{T}}(x,y) &= \tilde{\Phi}_s(y,x), \end{aligned}$$

where J_s the Jacobian of the stretching function, $J_s(x) = \det(DF_s(x))$. The functions $x \mapsto \tilde{\Phi}_s(x,y)$ and $x \mapsto \tilde{\Phi}_s^{\mathrm{T}}(x,y)$ satisfy the equations

$$(\tilde{\Delta}_s + k^2)\tilde{\Phi}_s(x,y) = -\delta(x-y), \tag{26}$$

$$(\tilde{\Delta}_s^{\mathrm{T}} + k^2)\tilde{\Phi}_s^{\mathrm{T}}(x,y) = -\delta(x-y), \tag{27}$$

where $\tilde{\Delta}_s^{\mathrm{T}}$ is the formal transpose of the operator $\tilde{\Delta}_s$,

$$\tilde{\Delta}_s^{\mathrm{T}} = \mathrm{div} H_s^{\mathrm{T}} H_s \mathrm{grad} + \mathrm{div} H_s^{\mathrm{T}} m_s.$$

Furthermore, if $\mathrm{Im}\, s = \beta > 0$, they satisfy the asymptotic decay conditions

$$\lim_{h(x)\to\infty} \sup\nolimits_{y\in K} e^{(\beta k - \varepsilon)\tau(h(x))} |D_x^\alpha \tilde{\Phi}_s(x,y)| = 0, \ |\alpha| \le 2 \tag{28}$$

$$\lim_{h(x)\to\infty} \sup\nolimits_{y\in K} e^{(\beta k - \varepsilon)\tau(h(x))} |D_x^\alpha \tilde{\Phi}_s^{\mathrm{T}}(x,y)| = 0, \ |\alpha| \le 1 \tag{29}$$

where $\varepsilon > 0$ is arbitrary, and $K \subset (\mathbb{R}^n \setminus \Omega)$ is any compact set.

Again, if $s = i$, we suppress the s–dependence.

In order to find a counterpart for the double surface operator P consider the exterior Dirichlet problem of finding an $u \in H^1_{(k-\varepsilon)}(\mathbb{R}^n \setminus \overline{D}_1)$ satisfying

$$(\tilde{\Delta} + k^2)u = 0 \text{ in } \mathbb{R}^n \setminus D_1, \tag{30}$$

$$u|_{\partial D_1} = f. \tag{31}$$

A natural candidate, analogously to the usual scattering theory, would be a solution given as

$$u = \tilde{K}_{\partial D_1, \mathbb{R}^n \setminus \overline{D}_1} \varphi,$$

where $\tilde{K}_{\partial D_1, \mathbb{R}^n \setminus \overline{D}_1}$ denotes the double layer potential defined by using the fundamental solution $\tilde{\Phi}$ as a kernel,

$$\tilde{K}_{\partial D_1, \mathbb{R}^n \setminus \overline{D}_1} \varphi(x) = \int_{\partial D_1} \partial_{c(y)} \tilde{\Phi}(x,y) \varphi(y) dS(y), \quad x \in \mathbb{R}^n \setminus \overline{D}_1. \tag{32}$$

Here, $\partial_{c(y)}$ denotes the conormal derivative associated to the operator $\tilde{\Delta}$,

$$\partial_{c(y)}\varphi(y) = n(y)^{\mathrm{T}} H(y)^{\mathrm{T}} H(y) \mathrm{grad}\varphi(y)|_{\partial D_1}. \tag{33}$$

As will be shown in Section 5, the double layer operator thus defined satisfies similar jump relations as the standard double layer operator. The function u thus defined satisfies the Dirichlet problem if φ satisfies $\varphi = (1/2+\tilde{K}_{\partial D_1})^{-1} f$, and the analogue of the operator P in (25) would be obtained by simply replacing K by $\tilde{K}$. However, the invertibility of the operator $1/2 + \tilde{K}_{\partial D_1}$ is not evident in this case. Indeed, by following the standard reasoning of scattering theory (see e.g. [7]), we are lead to consider the adjoint interior problem

$$\begin{aligned} (\tilde{\Delta}^{\mathrm{T}} + k^2)u &= 0 \text{ in } D_1, \\ \partial_c u|_{\partial D_1} &= h. \end{aligned}$$

The unique solvability of this problem is in general a non–trivial question. However, by using spectral theory, one can prove the following perturbation result.

Lemma 2.3 *For any $\varepsilon > 0$, there is an operator $A : L^2(D_1) \to L^2(D_1)$ of the form $Au = \sum_{j=1}^{N} (u, \alpha_j)\beta_j,\ \alpha_j, \beta_j \in C_0^\infty(D_1)$ such that $||A|| < \varepsilon$ and that the problem*

$$\begin{aligned} (\tilde{\Delta}^{\mathrm{T}} + A^{\mathrm{T}} + k^2)u &= 0 \textit{ in } D_1 \\ \partial_c u|_{\partial D_1} &= 0 \end{aligned}$$

has only trivial solution $u = 0$.

Notice that since the functions α_j and β_j are supported in D_1, the operators $\tilde{\Delta} + A$ and $\tilde{\Delta}$ coincide in $\mathbb{R}^n \setminus \overline{D}_1$ Therefore, in (30), we may replace $\tilde{\Delta}$ by $\tilde{\Delta} + A$ and consider the double layer operator corresponding to the perturbed operator. In Section 5, we construct the fundamental solution $\tilde{\Phi}_{(A)}$ satisfying

$$(\tilde{\Delta} + A + k^2)\tilde{\Phi}_{(A)}(x, y) = -\delta(x - y),$$

$$\lim_{h(x)\to\infty} \sup\nolimits_{y\in K} e^{(k-\varepsilon)\tau(h(x))} |D_x^\alpha \tilde{\Phi}_{(A)}(x, y)| = 0, \quad |\alpha| \le 2.$$

with any $\varepsilon > 0$, $K \in \mathbb{R}^n$ being any compact set. By obvious notations, we define now the perturbed layer operator $\tilde{K}_{(A)}$ by a formula similar to (32). By using this operator, we are able to prove the following result.

Lemma 2.4 *The problem (30)–(31) has a unique solution in* $H^1_{(1-\varepsilon)}(\mathbb{R}^n \setminus \overline{D}_1)$ *and it can be represented as* $u = K_{(A),\partial D_1,\mathbb{R}^n\setminus D_1}\varphi$, *where* φ *is the solution of*

$$(\frac{1}{2} + K_{(A),\partial D_1})\varphi = f. \tag{34}$$

From this result and the unique solvability of the full–space Bérenger problem, we can now deduce the following near–field formulation of the full–space problem, an analogue of Proposition 2.1

Theorem 2.4 *The restriction of the Bérenger solution* u_{B} *to the set* $D_2 \setminus \overline{\Omega}$ *is the unique solution in* $H^1(D_2 \setminus \overline{\Omega})$ *that satisfies the system*

$$(\tilde{\Delta} + k^2)u = 0 \text{ in } D_2 \setminus \overline{\Omega}, \tag{35}$$

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g \in H^{-1/2}(\partial\Omega), \tag{36}$$

$$u|_{\partial D_2} = P_{\mathrm{B}}(u|_{\partial D_1}), \tag{37}$$

where the double surface operator P_{B} *is given by the formula*

$$P_{\mathrm{B}} = \tilde{K}_{(A),\partial D_1,\partial D_2}(\frac{1}{2} + \tilde{K}_{(A),\partial D_1})^{-1}. \tag{38}$$

The proof of the Theorem A is now obtained by combining the previous results with the following two theorems. The first one claims that the truncation of the stretched exterior domain has little effect on P_{B}:

Theorem 2.5 *Let* D_1 *and* D_2 *be as before, and let* $\rho > \rho_2$. *There exists an operator* $P_\rho : H^{1/2}(\partial D_1) \to H^{1/2}(\partial D_2)$ *such that the truncated Bérenger problem (19)–(21) is equivalent to the near field problem*

$$(\tilde{\Delta} + k^2)u = 0 \text{ in } D(\rho) \setminus \overline{\Omega}, \tag{39}$$

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g, \tag{40}$$

$$u|_{\partial D_2} = P_\rho(u|_{\partial D_1}). \tag{41}$$

Moreover, we have

$$\lim_{\rho\to\infty} e^{(k-\varepsilon)\tau(\rho)}\|P_\rho - P_{\mathrm{B}}\| = 0$$

for any $\varepsilon > 0$.

The operator P_ρ is constructed in Section 6. Due to the technical nature, the construction is not summarized here. The final step is the following stability result.

Lemma 2.5 *Assume that* $\tilde{P} : H^{1/2}(\partial D_1) \to H^{1/2}(\partial D_2)$ *is an operator with the property*

$$\|\tilde{P} - P_{\rm B}\| < \varepsilon,$$

the norm being the uniform operator norm of the space of bounded linear operators $H^{1/2}(\partial D_1) \to H^{1/2}(\partial D_2)$. *Consider the system (35)–(37) with* P *replaced by* $\tilde{P}$. *For* $\varepsilon > 0$ *small enough, the system has a unique solution* $\tilde{u} \in H^1(D_2 \setminus \overline{\Omega})$, *and we have*

$$\|u_{\rm B} - \tilde{u}\|_{H^1(D_2\setminus\overline{\Omega})} < C\varepsilon$$

for some positive constant C.

The proof of this stability result is the same as in [13] (Lemma 2.2). Therefore, the proof is omitted in this paper.

As a concluding remark, we mention that we have included an appendix where the explicit form of the Bérenger equation using the tangent–normal coordinate system has been derived. This calculation shows the connection to the previous works done in circular coordinate systems, and may be helpful in the implementation of the PML equation by the finite element method.

3 Geometric considerations

The remaining part of the paper is devoted to detailed discussion of the results in the previous section. We begin with some simple geometric results.

Let $D \subset \mathbb{R}^n$ denote the bounded domain, $\overline{\Omega} \subset D$, where the numerical approximation of the scattering solution is required. We assume that D is stricly convex and ∂D is a C^2–surface. The functions $h(x)$ and $p(x)$, $x \in \mathbb{R}^n \setminus D$ are defined by the equation (4).

Let $x_0 \in \partial D$ and denote by $B_\varepsilon(x_0)$ a ball with radius $\varepsilon > 0$ centered at x_0. If ε is chosen small enough, we can parameterize the surface patch $V_\varepsilon(x_0) = B_\varepsilon(x_0) \cap \partial D$ by projecting it to the tangent of ∂D through x_0. Let us denote by $D_\perp n(x_0) \in \mathbb{R}^{(n-1)\times(n-1)}$ the differential of the normal vector with respect to the tangential components of x at x_0. If the surface patch $V_\varepsilon(x_0)$ is given in the form $x_n = \varphi(x_1, \dots, x_{n-1})$, $\varphi \in C^2$, we have

$$D_\perp n(x_0) = \left(\frac{\partial^2 \varphi}{\partial x_j \partial x_k} \right)_{1\le j,k\le n-1}.$$

The eigenvalues of the matrix $-D_\perp n(x_0)$ are the principal curvatures of the surface ∂D at x_0, denoted here as $c_j(x_0)$, $1 \le j \le n-1$. In the sequel, we make the extra assumption that the principal curvatures are strictly positive.

Let a denote the function defined in Definition 2.1. By a straightforward differentiation, we have

$$Da(x) = \tau'(h(x))n(x)n(x)^{\mathrm{T}} + \tau(h(x))Dn(x), \tag{42}$$

where we used the identity $Dh(x) = n(x)^{\mathrm{T}}$. From this representation, we obtain the following result.

Lemma 3.1 *The matrix $Da(x)$ is symmetric and the eigenvalues are*

$$\lambda_j(x) = \tau(h(x))\frac{c_j(x)}{1 + c_j(x)h(x)}, \; 1 \le j \le n-1, \quad \lambda_n(x) = \tau'(h(x)),$$

where the numbers $c_j(x)$ are the principal curvatures of ∂D at $p(x) \in \partial D$.

Proof: The eigenvalue $\tau'(h(x))$ that corresponds to the eigenvector $n(x)$ can be read off immediately from the representation (42). To find the other eigenvalues, denote by ∂D_h the inflated surface

$$\partial D_h = \{x = p + hn(p) \mid p \in \partial D\}, \quad h > 0.$$

For $x \in \partial D_h$, fix the local coordinates such that x_n–axis is parallel to n. In this coordinate system, we have

$$Dn(x) = \begin{pmatrix} D_\perp n(x) & 0 \\ 0 & 0 \end{pmatrix}.$$

From this representation, it is evident that the eigenvalues of $Dn(x)$ equal to the negative of the principal curvatures of the inflated surface at x. In order to find these numbers, consider the one–to–one mapping $\partial D \to \partial D_h$, $p \mapsto p + hn(p)$. Clearly, $n(x(p)) = n(p)$. Therefore,

$$D_p n(p) = D_p(n(x(p))) = D_x n(x(p))(I + hD_p n(p)).$$

Applying both sides of this identity to the eigenvectors of $D_p n(p)$ the claim follows. $\square$

Corollary 3.1 *The Jacobian J_s of the stretching function F_s is given as*

$$J_s(x) = \det(F_s(x)) = b_1(x,s)\ldots b_n(x,s),$$

where

$$b_j(x,s) = 1 + s\lambda_j(x), \quad 1 \le j \le n. \tag{43}$$

In the following lemma, we use the notation $\tilde{x} = F_s(x) \in \Gamma_s$, where F_s is the stretching function of Definition 2.1.

Lemma 3.2 *Let $s = \alpha + i\beta$ be the stretching parameter, $\alpha \ge 0$, $\beta \ge 0$. Let $\tilde{x}_j = \tilde{x}(x_j)$, $j = 1, 2$, where $x_j \in \mathbb{R}^n \setminus D$, and denote the representations (4) of x_j as*

$$x_j = p_j + h_j n_j, \quad j = 1, 2. \tag{44}$$

We have the estimate

$$\begin{aligned} \mathrm{Im}\,(\tilde{x}_1 - \tilde{x}_2)^2 \ge 2\beta(C(\tau_1 + \tau_2)|p_1 - p_2|^2 \;&+\; |h_1 - h_2||\tau_1 - \tau_2| \\ &+\; \alpha|\tau_1 n_1 - \tau_2 n_2|^2), \quad \tau_j = \tau(h_j), \end{aligned}$$

where $C > 0$ is a constant depending only on the geometry of the surface ∂D.

Proof: The assumption that principal curvatures are strictly positive implies in particular that for all $x, y \in \mathbb{R}^n \setminus D$,

$$(p(x) - p(y))^{\mathrm{T}} n(x) \ge C|p(x) - p(y)|^2 \tag{45}$$

for some constant $C > 0$ depending only on the geometry of the surface.

By definition, we have

$$\mathrm{Im}(\tilde{x}_1 - \tilde{x}_2)^2 = 2\beta((x_1 - x_2)^{\mathrm{T}}(\tau_1 n_1 - \tau_2 n_2) + \alpha|\tau_1 n_1 - \tau_2 n_2|^2).$$

By substituting the representations (44) and rearranging the terms, we obtain

$$\begin{aligned}(x_1 - x_2)^{\mathrm{T}}(\tau_1 n_1 - \tau_2 n_2) &= \tau_1(s_1 - s_2)^{\mathrm{T}} n_1 + \tau_2(s_2 - s_1)^{\mathrm{T}} n_2 \\ &+ h_1\tau_1 + h_2\tau_2 - (h_1\tau_2 + h_2\tau_1) n_1^{\mathrm{T}} n_2.\end{aligned}$$

Here,

$$h_1\tau_1 + h_2\tau_2 - (h_1\tau_2 + h_2\tau_1) n_1^{\mathrm{T}} n_2 \geq (h_1 - h_2)(\tau_1 - \tau_2) \geq 0,$$

since the function τ is strictly increasing. Further, since D is strictly convex, we have $(s_1 - s_2)^{\mathrm{T}} n_1 \geq 0$ and $(s_2 - s_1)^{\mathrm{T}} n_2 \geq 0$. The claim follows now from the estimate (45). □

The estimate of the previous lemma allows us to prove Lemma 2.1.

Proof of Lemma 2.1: If $\tilde{x} = \tilde{x}(x) \in \Gamma \setminus \overline{\Omega}$ and $y \in \partial\Omega$, we see as in Lemma 3.2 that

$$\operatorname{Im}(\tilde{x}(x) - y)^2 \geq 2C\tau(h(x))|p(x) - y|^2 + 2h(x)\tau(h(x)) > 0,$$

i.e., $\tilde{x} - y \in G$. Since G is open, we may choose neighborhoods $U(\tilde{x}, y) \subset \mathbb{C}^n$ of $\tilde{x}$ and $V(\tilde{x}, y) \subset \partial\Omega$ of y such that $U(\tilde{x}, y) \times V(\tilde{x}, y) \subset G$. Since $\{\tilde{x}\} \times \partial\Omega$ is compact, we can pick an open finite sub-cover such that

$$\{\tilde{x}\} \times \partial\Omega \subset \bigcup_{j=1}^{n} U(\tilde{x}, y_j) \times V(\tilde{x}, y_j).$$

Let us denote $\tilde{U}(\tilde{x}) = \cap_{j=1}^{n} U(\tilde{x}, y_j)$. The claim follows by choosing now

$$U = \bigcup_{\tilde{x} \in \Gamma \setminus \overline{\Omega}} \tilde{U}(\tilde{x}).$$

□

4 Bérenger operator

Here we give the details of the derivation of the operator $\tilde{\Delta}_s$. Also, we prove the ellipticity properties of this operator.

Before deriving the formula of $\tilde{\Delta}_s$ in real coordinates, we give the proof of Lemma 2.2.

Proof of Lemma 2.2: The claims (i) and (ii) are obvious by the construction, so we need only to show (iii).

It is a straightforward matter to check that

$$(\Delta_z + k^2)\Phi(z,y)|_{z=x\in\mathbb{R}^n} = (\Delta_x + k^2)\Phi(x,y) = 0, \ x \neq y,$$

and the same holds for $z \mapsto \partial\Phi(z,y)/\partial n(y)$. Thus, we have

$$(\Delta_z + k^2)u(z,s)|_{z\in D\setminus\overline{\Omega}} = 0.$$

Let $B \subset U$ be a small polydisc with the center $x \in D \setminus \overline{\Omega}$. Since $(\Delta_z + k^2)u(z,s)|_{B\cap\mathbb{R}^n} = 0$, it follows by analyticity that $(\Delta_z + k^2)u(z,s)|_B = 0$. Since U is connected, this implies that $(\Delta_z + k^2)u(z,s) = 0$ in B. □

Now we derive the representation in real coordinates. Let $v : \mathbb{C}^n \to \mathbb{C}$ and $V : \mathbb{C}^n \to \mathbb{C}^n$ be analytic functions. We define the extensions of the gradient and divergence in $\mathbb{C}^n$ as

$$\begin{aligned} \mathrm{grad}_z v &= (\partial_{z_1} v, \ldots, \partial_{z_n} v)^{\mathrm{T}}, \\ \mathrm{div}_z V &= \sum_{j=1}^{n} \partial_{z_j} V_j. \end{aligned}$$

Note that $\mathrm{div}_z \mathrm{grad}_z = \Delta_z$ as usual. Similarly, the differential of V is defined as a matrix in $\mathbb{C}^{n\times n}$ with elements

$$(D_z V)_{i,j} = \partial_{z_j} V_i, \ 1 \leq i,j \leq n.$$

We prove the following simple result.

Lemma 4.1 *Let $F : \mathbb{R}^n \to \mathbb{C}^n$ be a differentiable immersion. Then we have*

$$\begin{aligned} \mathrm{grad}_z v \circ F &= H \mathrm{grad}_x (v \circ F), \\ \mathrm{div}_z V \circ F &= \mathrm{div}_x (H^{\mathrm{T}} V \circ F) - m^{\mathrm{T}} V \circ F, \end{aligned}$$

where $H = (DF(x)^{\mathrm{T}})^{-1} \in \mathbb{C}^{n\times n}$ and $m \in \mathbb{C}^n$,

$$m_j = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} H_{j,k}.$$

Proof: For analytic functions, the normal chain rules apply, so

$$\mathrm{grad}_x(v \circ F) = (DF)^{\mathrm{T}} \mathrm{grad}_z v \circ F,$$

and the first claim follows.

For the second identity, we write first by the chain rule

$$D(V \circ F) = (D_z V \circ F) DF,$$

and thus

$$\mathrm{div}_z V \circ F = \mathrm{Tr}(D_z V \circ F) = \mathrm{Tr}(D(V \circ F) H^{\mathrm{T}}),$$

and further, since

$$\begin{aligned} (D(V \circ F)H^{\mathrm{T}})_{i,j} &= \sum_{k=1}^{n} \frac{\partial}{\partial x_k}(V_i \circ F) H_{j,k} \\ &= \sum_{k=1}^{n} \frac{\partial}{\partial x_k}((V_i \circ F) H_{j,k}) - \sum_{k=1}^{n} V_i \circ F \frac{\partial}{\partial x_k} H_{j,k}, \end{aligned}$$

so applying the trace the claim follows. □

As a corollary, we obtain immediately the claim of Theorem 2.1.

From the representation in terms of real variables, we obtain the following ellipticity result.

Lemma 4.2 *There exists a function $\alpha : \mathbb{R}^n \setminus \overline{\Omega} \to \mathbb{C}$ such that the operator $\alpha\tilde{\Delta}$ is elliptic. Moreover, for $\tilde{\Delta}$ restricted to any bounded open set A, we can choose α=constant in A.*

Proof: By definition, it is enough to show that there exists such an α that

$$-\mathrm{Re}\Big(\alpha(x)\overline{z}^{\mathrm{T}} H(x)^{\mathrm{T}} H(x) z\Big) > 0$$

for $z \in \mathbb{C}^n$, $z \neq 0$. Observe that here H^{T} is the real transpose of the complex matrix H without complex conjugation. The matrix $H = (I + iDa^{\mathrm{T}})^{-1}$ is invertible. Hence we see by substituting $z = (H^{\mathrm{T}} H)^{-1}\zeta$ that is is enough to find such an α that

$$-\mathrm{Re}\Big(\alpha\overline{\zeta}^{\mathrm{T}}(I - iDa)^{\mathrm{T}}(I - iDa)\zeta\Big) > 0, \ \zeta \neq 0.$$

By writing $\zeta = \zeta_R + i\zeta_I$, $\zeta_R, \zeta_I \in \mathbb{R}^n$, we have

$$\begin{aligned}\overline{\zeta}^{\mathrm{T}}(I - iDa)^{\mathrm{T}}(I - iDa)\zeta &= |\zeta|^2 - |Da\zeta|^2 - 2i(\zeta_R^{\mathrm{T}} Da\zeta_R + \zeta_I^{\mathrm{T}} Da\zeta_I) \\ &= \Big(|\zeta_R|^2 - |Da\zeta_R|^2 - 2i\zeta_R^{\mathrm{T}} Da\zeta_R\Big) \\ &\quad + \Big(|\zeta_I|^2 - |Da\zeta_I|^2 - 2i\zeta_I^{\mathrm{T}} Da\zeta_I\Big).\end{aligned}$$

Next we the diagonalization $Da = V\mathrm{diag}(\lambda_1, \dots, \lambda_n)V^{\mathrm{T}}$ and denote $V^{\mathrm{T}}\zeta_R = |\zeta_R|\omega$, $V^{\mathrm{T}}\zeta_I = |\zeta_R|\eta$, where $\omega, \eta \in \mathbb{S}^{n-1}$, to obtain

$$\overline{\zeta}^{\mathrm{T}}(I - iDa)^{\mathrm{T}}(I - iDa)\zeta = |\zeta_R|^2 \sum_{j=1}^{n}(1 - i\lambda_j)^2\omega_j^2 + |\zeta_I|^2 \sum_{j=1}^{n}(1 - i\lambda_j)^2\eta_j^2.$$

From this representation we see that the complex numbers $\sum(1 - i\lambda_j)^2\omega_j^2$ and $\sum(1 - i\lambda_j)^2\eta_j^2$ lie within the convex hull (e.g. line segment for $n = 2$ and triangle for $n = 3$) of the points $(1 - i\lambda_j)^2$. By choosing $\alpha = \alpha(x)$ in a proper way such that this set is rotated to the half plane $\mathrm{Re}\, z < 0$ by multiplication of α we get the claim. For a bounded set A we note that the union of the convex hulls over $x \in \overline{A}$ form a compact set in $\{z \in C \mid -\pi/2 < \arg z \leq 0\}$. $\square$

In the following lemma, we use again the convention $J(x) = J_s(x)$ for $s = i$, where J_s is the Jacobian given in Corollary 3.1. We use also the notation

$$j_s(x) = b_1(x, s) \dots b_{n-1}(x, s),$$

and $j(x) = j_i(x)$. By Lemma 3.1 and the assumed growth properties of the functions τ and τ', we have

$$\lim_{h(x)\to\infty} \arg\left(b_j(x)\right) = \frac{\pi}{2}.$$

We say that $\rho > 0$ is *in the exponential range* if the following condition holds: For $x \in \mathbb{R}^n$, $h(x) > \rho$, the inequality $|\arg\left(-ib_j(x)\right)| < \delta$ hold for all j, $1 \leq j \leq n$, and some δ, $0 < \delta < \pi/2n$.

Lemma 4.3 *Let $A = \{x \in \mathbb{R}^n \mid \rho_1 < h(x) < \rho_2\}$, where ρ_1 is in the exponential range. Then for all $\varphi \in H^1(A)$, we have the coercivity estimate*

$$\begin{aligned}\mathrm{Re}(-i)^n\Big(\int_A (\tilde{\Delta} + k^2)\varphi\,\overline{\varphi}J dx &- \int_{\partial A}(n^{\mathrm{T}} H \mathrm{grad}\varphi)\overline{\varphi} j dS\Big) \\ &\geq \min(C_A, k^2)\cos(n\delta)\|\varphi\|_{H^1(A)}^2,\end{aligned}$$

where $C_A = \inf\{|b_j(x)|^{-1} \mid x \in A,\ 1 \le j \le n\}$.

Proof: Let $s > 0$, whence $F_s : \mathbb{R}^n \to \mathbb{R}^n$ is C^2-diffeomorphism. An application of Green's formula together with equation (15) we have

$$\begin{aligned}
\int_A (\tilde{\Delta}_s + k^2)\varphi\,\overline{\varphi} J_s dx &= \int_{F_s(A)} (\Delta + k^2)[\varphi \circ F_s^{-1}]\overline{\varphi} \circ F_s^{-1} dx \\
&= \int_{F_s(A)} (-(\mathrm{grad}[\varphi \circ F_s^{-1}]^{\mathrm{T}}\mathrm{grad}[\overline{\varphi} \circ F_s^{-1}] + k^2|\varphi \circ F_s^{-1}|^2)dx \\
&+ \int_{\partial F_s(A)} n^{\mathrm{T}}\mathrm{grad}[\varphi \circ F_s^{-1})]\overline{\varphi} \circ F_s^{-1} dS.
\end{aligned}$$

Next, we transform the integrals back to the original coordinates and note that the Jacobian of the change of the variables at the boundary ∂A is j_s and that the normal vector at the boundary is unaltered. Therefore,

$$\begin{aligned}
\int_A (\tilde{\Delta}_s + k^2)\varphi\,\overline{\varphi} J_s dx &= \int_A (-(H_s\mathrm{grad}\varphi)^{\mathrm{T}}(H_s\mathrm{grad}\overline{\varphi}) + k^2|\varphi|^2)J_s dx \\
&+ \int_{\partial A} n^{\mathrm{T}} H_s \mathrm{grad}\varphi\overline{\varphi} j_s dS.
\end{aligned}$$

Since above both sides depend analytically on s, we can continue this equation to $s = i$. (Here as in the sequel we use without further mentioning the fact that the s–dependent functions allow an analytic extension to a neighborhood of the real axis.)

To obtain the desired formula, denote by α_j, $1 \le j \le n$, the orthonormal eigenvectors of the symmetric matrix Da. We have $H\alpha_j = b_j^{-1}\alpha_j$. Further, by writing $\mathrm{grad}\varphi$ in this basis as $\mathrm{grad}\varphi = \sum \xi_j \alpha_j$, we have

$$\begin{aligned}
&(-i)^n(-(H\mathrm{grad}\varphi)^{\mathrm{T}}(H\mathrm{grad}\overline{\varphi}) + k^2|\varphi|^2)J \\
&\qquad = \sum_{j=1}^n |\xi_j|^2 \frac{(-ib_1)\ldots(-ib_n)}{(-ib_j)^2} + k^2|\varphi|^2(-ib_1)\ldots(-ib_n).
\end{aligned}$$

By the assumptions, we have

$$\left|\arg\left(\frac{(-ib_1)\ldots(-ib_n)}{(-ib_j)^2}\right)\right| = |\sum_{\ell=1}^n \arg\,(-ib_\ell) - 2\arg\,(-ib_j)| \le n\delta,$$

and similarly

$$|\arg\,(-ib_1)\dots(-ib_n))| = |\sum_{\ell=1}^{n} \arg\,(-ib_\ell)| \leq n\delta.$$

Furthermore, since $|b_j| > 1$, we have

$$\left|\frac{(-ib_1)\dots(-ib_n)}{(-ib_j)^2}\right| \geq \inf_{x\in A, 1\leq j\leq n} \frac{1}{|b_j(x)|} = C_A.$$

From these estimates, the claim follows. □

We conclude this section by proving the perturbation result, Lemma 2.3.

Proof of Lemma 2.3: In this proof the operator $L = (\tilde{\Delta}^{\mathrm{T}} + k^2)$ is interpreted as a closed unbounded operator in $L^2(D_1)$ with domain $\mathcal{D}(L) = \{u \in H^2(D_1) \mid \partial_c u|_{\partial D_1} = 0\}$. Since L is elliptic, its spectrum consists of discrete eigenvalues with finite algebraic multiplicities. If zero is not an eigenvalue, the claim is true with $A = 0$. Otherwise, let Σ be a smooth contour in the resolvent set of L enclosing only zero from the spectrum. We define the Riesz projection

$$P = -\frac{1}{2\pi i}\int_\Sigma (L - z)^{-1}\,dz.$$

By [10], P is a finite dimensional projection that defines an L–invariant decomposition $L^2(D_1) = \mathrm{Ran}\,(P) \oplus \mathrm{Ran}\,(I - P)$. Moreover, the restriction of L in $\mathrm{Ran}\,(I - P)$ is invertible and the spectrum of the restriction of L in $\mathrm{Ran}\,(P)$ is the set $\{0\}$. Thus we see that with arbitrarily small $t > 0$ the operator $L + tP$ is invertible. Since P can be represented as $Pu = \sum(u, \varphi_j)\psi_j$ with some $\varphi_j, \psi_j \in L^2(D_1)$, we get the claim by approximating the functions φ_j and ψ_j with $C_0^\infty(D_1)$–functions α_j and β_j in $L^2(D_1)$-norm. □

5 Green's functions and layer potentials

In this section, we discuss the construction of the various fundamental solutions associated to the Bérenger equation. We start with the proof of Theorem 2.3.

Proof of Theorem 2.3: The claim $F_s(x) - F_s(y) \in G$ follows from the Lemma 3.2. We start by proving the equation (27). Again, let $s > 0$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$. By the formula (15), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \tilde{\Phi}_s^{\mathrm{T}}(x,y)(\tilde{\Delta}_s + k^2)\varphi(x)dx \\
&\qquad = \int_{\mathbb{R}^n} \Phi(F_s(y), F_s(x))(\Delta + k^2)[\varphi \circ F_s^{-1}](F_s(x))J_s(x)dx \\
&\qquad = \int_{\mathbb{R}^n} \Phi(F_s(y), x)(\Delta + k^2)[\varphi \circ F_s^{-1}](x)dx \\
&\qquad = -\varphi(y)
\end{aligned}$$

by a change of variables. Since the right side is independent of s while the left side is analytic in s, the equation holds in the whole $\overline{\mathbb{C}}^{++}$, and the claim (27) follows.

In order to show (26), we observe first that for any test functions $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ and $s > 0$, we have, by using formula (15) and Green's formula, the identity

$$\int_{\mathbb{R}^n} [(\tilde{\Delta}_s + k^2)\varphi]\psi J_s dx = \int_{\mathbb{R}^n} \varphi[(\tilde{\Delta}_s + k^2)\psi]J_s dx,$$

or

$$\begin{aligned}
\tilde{\Delta}_s + k^2 &= (J_s(x)(\tilde{\Delta}_s + k^2)J_s(x)^{-1})^{\mathrm{T}} \\
&= J_s(x)^{-1}(\tilde{\Delta}_s + k^2)^{\mathrm{T}} J_s(x), \qquad (46)
\end{aligned}$$

and again by analyticity, this formula is valid for $s \in \overline{\mathbb{C}}^{++}$. On the other hand, we have

$$\begin{aligned}
\tilde{\Phi}_s(x,y) &= J_s(y)\Phi(F_s(x), F_s(y)) \\
&= J_s(x)^{-1}\tilde{\Phi}_s^{\mathrm{T}}(x,y)J_s(y).
\end{aligned}$$

Hence, we get

$$(\tilde{\Delta}_s + k^2)\tilde{\Phi}_s(x,y) = J_s(x)^{-1}(\tilde{\Delta}_s + k^2)^{\mathrm{T}}\tilde{\Phi}_s^{\mathrm{T}}(x,y)J_s(y) = -\delta(x-y).$$

To prove the decay estimates (28) and (29), let us write $s = \alpha + i\beta$, $\alpha \geq 0$, $\beta > 0$, and for brevity $h = h(x)$ and $\tau = \tau(h(x))$. By the definition of the stretching function F_s, we easily get the estimates

$$\begin{aligned}\operatorname{Re}\,(F_s(x)-F_s(y))^2 &= (\alpha^2-\beta^2)\tau^2(1+\varepsilon_1),\\ \operatorname{Im}\,(F_s(x)-F_s(y))^2 &= 2\beta\tau^2(\alpha+\varepsilon_2),\end{aligned}$$

where ε_1 and ε_2 depend analytically on α and β and they have the property

$$\lim_{h\to\infty}\mathrm{sup}_{y\in K}\varepsilon_j = 0, \quad j=1,2.$$

From these representations, we obtain further that

$$\operatorname{Im}\,((F_s(x)-F_s(y))^2)^{1/2} = \beta\tau(1+\varepsilon_3),$$

where ε_3 shares the properties of ε_1 and ε_2. The claim (28) follows now from the asymptotic estimate of Hankel functions,

$$|D^j H^{(1)}_{(n-2)/2}(z)| \leq C\frac{1}{\sqrt{|z|}}e^{-i\operatorname{Im} z}, \quad j\leq 2,$$

as $|z|\to\infty$, and the boundedness of $J_s(y)$ in a compact set.

Similarly, we see that the growth conditions (6) imposed on the derivatives of the function τ restrict the growth of $|D^\alpha J_s(x)|$, $|\alpha|\leq 1$, and the estimate (29) follows. □

In the construction of Bérenger solutions, we need a counterpart of the classical Stratton–Chu representation formulas, given in the following lemma.

Lemma 5.1 *Let $G\subset\mathbb{R}^n$ be bounded domain or a complement of a bounded domain with a C^2–boundary. Assume that u satisfies the equation $(\tilde{\Delta}+k^2)u = 0$ in G and, if G is unbounded, that u vanishes exponentially at infinity. Then u admits a representation*

$$u(x) = \pm\int_{\partial G}(\tilde{\Phi}(x,y)\partial_{c(y)}u(y) - u(y)(\partial_c - b(y))\tilde{\Phi}(x,y))dS(y), \quad x\in G, \quad (47)$$

where the plus sign corresponds to the case of a bounded domain, the minus sign to the exterior domain. Above, ∂_c denotes the conormal derivative (33) at the boundary and

$$b(y) = n(y)^{\mathrm{T}}H(y)^{\mathrm{T}}m(y).$$

Proof: Assume first that G is bounded. By denoting briefly $H = H(y)$ and $m = m(y)$, we have

$$\begin{aligned}
u(x) &= -\int_G ((\tilde{\Delta}+k^2)^{\mathrm{T}}\tilde{\Phi}^{\mathrm{T}}(y,x)u(y) - \tilde{\Phi}^{\mathrm{T}}(y,x)(\tilde{\Delta}+k^2)u(y))dy \\
&= -\int_G (\mathrm{div}H^{\mathrm{T}}H\mathrm{grad}\tilde{\Phi}^{\mathrm{T}}(y,x)u(y) - \tilde{\Phi}^{\mathrm{T}}(y,x)\mathrm{div}H^{\mathrm{T}}H\mathrm{grad}u(y))dy \\
&\qquad + \int_G (m^{\mathrm{T}}H\mathrm{grad}\tilde{\Phi}^{\mathrm{T}}(y,x)u(y) + \tilde{\Phi}^{\mathrm{T}}(y,x)\mathrm{div}(H^{\mathrm{T}}mu(y)))dy \\
&= -\int_{\partial G} n^{\mathrm{T}}\Big(H^{\mathrm{T}}H\mathrm{grad}\tilde{\Phi}^{\mathrm{T}}(y,x)u(y) \\
&\qquad - \tilde{\Phi}^{\mathrm{T}}(y,x)H^{\mathrm{T}}H\mathrm{grad}u(y) - H^{\mathrm{T}}m\tilde{\Phi}^{\mathrm{T}}(y,x)u(y)\Big)dS(y).
\end{aligned}$$

Since $\tilde{\Phi}^{\mathrm{T}}(y,x) = \tilde{\Phi}(x,y)$ this gives the integral representation.

For an unbounded domain, we apply in the usual manner the above formula first in $B_R \setminus \overline{G}$, B_R being a ball with radius R. Since u and Green's function vanish exponentially, the integral over ∂B_R tends to zero when $R \to \infty$. $\square$

Remark: If G has the form $G = D(\rho) = \{x \in \mathbb{R}^n \mid h(x) < \rho\}$, then $\partial_c = (1+i\tau'(h(x)))^{-2}\partial/\partial n$. Indeed, in this case the normal n is an eigenvector of $H(x)$ with eigenvalue $b_n(x)^{-1}$.

As in the standard scattering theory, we define now the Bérenger single and double layer potentials as follows: If $X \subset \mathbb{R}^n$, $X \neq \emptyset$ and $\partial G \cap X = \emptyset$, we use the notation

$$\begin{aligned}
\tilde{S}_{\partial G,X}\varphi(x) &= \int_{\partial G} \tilde{\Phi}(x,y)\varphi(y)dS(y), \quad x \in X, \\
\tilde{K}_{\partial G,X}\psi(x) &= \int_{\partial G} \partial_{c(y)}\tilde{\Phi}(x,y)\psi(y)dS(y), \quad x \in X.
\end{aligned}$$

We use also the transposed operators

$$\begin{aligned}
\tilde{S}^{\mathrm{T}}_{\partial G,X}\varphi(x) &= \int_{\partial G} \tilde{\Phi}^{\mathrm{T}}(x,y)\varphi(y)dS(y), \quad x \in X, \\
\tilde{K}^{\mathrm{T}}_{\partial G,X}\psi(x) &= \int_{\partial G} \partial_{c(x)}\tilde{\Phi}^{\mathrm{T}}(x,y)\psi(y)dS(y), \quad x \in X.
\end{aligned}$$

Observe that $\partial_c \tilde{S}^{\mathrm{T}} = \tilde{K}^{\mathrm{T}}$. The extensions (9)–(11) are special cases of these extensions, since at $\partial\Omega$, $\gamma(y) = 1$ and $\partial_c = \partial/\partial n$.

These layer potentials share many of the properties of the standard layer potentials of scattering theory. To show that they obey jump relations of the usual form, we need an auxiliary elementary result.

Lemma 5.2 *Assume that $A \subset \mathbb{C}$ is a connected domain which subset $S \subset A$ has a limit point in A. Let $f_n : A \to \mathbb{C}$ be analytic functions which are uniformly bounded in A and for which $\lim_{n\to\infty} f_n(s)$ exists for all $s \in S$. Then there is a unique analytic function $f : A \to \mathbb{C}$ such that $\lim_{n\to\infty} f_n(a) = f(a)$ for all $a \in A$.*

Proof. Since $\{f_n\}$ is uniformly bounded, it is a normal family ([1]), and hence there is a subsequence $\{f_{n_k}\}$ converging uniformly in compact subsets of A towars an analytic function $f : A \to \mathbb{C}$. In particular, f satisfies $f_n(s) \to f(s)$ for all $s \in S$.

Assume now that there is $a \in A$ for which $\lim f_n(a) = f(a)$ is not true. Then we can choose a subsequence $\{f_{m_k}\}$ of $\{f_n\}$ for which $f_{m_k}(a) \to h \neq f(a)$. Since also $\{f_{m_k}\}$ is a normal family there is a subsequence converging uniformly in compact subsets of A to an analytic function $g : A \to \mathbb{C}$. Since $f_n(s) \to f(s)$ for all $s \in S$, we must have $f(s) = g(s)$ for $s \in S$. Since S has a limit point, this yields $f = g$ in A which is a contradiction with the assumption $g(a) = h \neq f(a)$. □

Lemma 5.3 *Assume that X is a measurable set and $A \subset \mathbb{C}$. If $u \in L^1(X \times A)$ has the property that $z \mapsto u(x, z)$ is analytic for a.e. $x \in X$ and that there is $h \in L^1(X)$ such that $|u(x, z)| \leq h(x)$ for a.e. x and z. Then $v(z) = \int_X u(x, z)dx$ is analytic.*

Proof. By using Cauchy's and Fubini's theorems we see for a closed contour $\gamma \subset A$

$$\int_\gamma v(z)\, dz = \int_\gamma \int_X u(x, z)\, dxdz = \int_X \int_\gamma u(x, z)\, dzdx = 0.$$

By Lebesgue Theorem of Dominated Convergence, $v(z)$ is continuous. Hence the above with Morera's theorem yield that v is analytic. □

Now we are ready to prove the jump relations. Note, however, that the following result is deals only with domains obtained by inflating the original convex body D.

Theorem 5.1 *Let $G = D(\rho) = \{x \in \mathbb{R}^n \mid h(x) < \rho\}$ for some $\rho > 0$. The Bérenger layer potentials satisfy the jump relations*

$$\tilde{S}_{\partial G,G}\varphi(x)|_{\partial G}^{-} = \tilde{S}_{\partial G,\mathbb{R}^n\setminus\overline{G}}\varphi(x)|_{\partial G}^{+} = \tilde{S}_{\partial G}\varphi(x)$$

and the same is true for $\tilde{S}^{\mathrm{T}}$. For the double layer operators we have

$$\tilde{K}_{\partial G,G}\varphi(x)|_{\partial G}^{-} + \frac{1}{2}\varphi(x) = \tilde{K}_{\partial G,\mathbb{R}^n\setminus\overline{G}}\varphi(x)|_{\partial G}^{+} - \frac{1}{2}\varphi(x) = \tilde{K}_{\partial G}\varphi(x),$$

and

$$\tilde{K}^{\mathrm{T}}_{\partial G,G}\varphi(x)|_{\partial G}^{-} - \frac{1}{2}\varphi(x) = \tilde{K}^{\mathrm{T}}_{\partial G,\mathbb{R}^n\setminus\overline{G}}\varphi(x)|_{\partial G}^{+} + \frac{1}{2}\varphi(x) = \tilde{K}^{\mathrm{T}}_{\partial G}\varphi(x).$$

Here the notation $\cdot|_{\partial G}^{\pm}$ means the trace from inside or outside of ∂G.

Proof: For the single layer operators, the claim follows from the fact that the kernels $\tilde{\Phi}_s(x,y)$ and $\tilde{\Phi}_s^{\mathrm{T}}(x,y)$ are weakly singular and therefore they define continuous functions throughout $\mathbb{R}^n$ (see [7]).

Next consider $\tilde{K}$. For the operator $\tilde{K}^{\mathrm{T}}$, the proof is similar. As in the standard scattering theory (see [7]), it suffices to show the claim for $\varphi = 1$ and for the wave number $k = 0$. Consider first the double layer potential with a general stretching parameter s. By denoting $\tilde{\Phi}_s^0$ Green's function with the wave number $k = 0$, we have for $\varphi = 1$

$$\begin{aligned}
\tilde{K}_{\partial G,G}\varphi(x,s) &= \int_{\partial D(\rho)} \partial_{c(y)}\tilde{\Phi}_s^0(x,y)dS(y) \\
&= \int_{\partial D(\rho)} \frac{1}{b_n(y,s)^2} n(y)^{\mathrm{T}}\mathrm{grad}_y(j_s(y)b_n(y,s)\Phi^0(F_s(x),F_s(y)))dS(y),
\end{aligned}$$

where we wrote $J_s(y) = j_s(y)b_n(y,s)$. Since Φ^0 is weakly singular, it suffices to consider the most singular part when the gradient acts on Φ^0. Let us denote

$$u(x,s) = \int_{\partial D(\rho)} \frac{j_s(y)}{b_n(y,s)} n(y)^{\mathrm{T}}\mathrm{grad}_y\Phi^0(F_s(x),F_s(y))dS(y).$$

Assume first that $s > 0$. Then F_s is a diffeomorphism

$$F_s : \partial D(\rho) \to \partial D(\rho + s\tau(\rho))$$

and we have

$$j_s(y,s)dS(y) = dS(F_s(y)), \quad \frac{1}{b_n(y,s)} n(y)^{\mathrm{T}}\mathrm{grad}_y = n(F_s(y))^{\mathrm{T}}\mathrm{grad}_{F_s(y)},$$

so we get by a change of variables

$$u(x,s) = \int_{\partial D(\rho+s\tau(\rho))} n(y)^{\mathrm{T}}\mathrm{grad}_y \Phi^0(F_s(x),y)dS(y), \quad F_s(x) \in D(\rho + s\tau(\rho)).$$

As x approaches the boundary $\partial D(\rho)$, $F_s(x)$ approaches $\partial D(\rho + s\tau(\rho))$, and so for real stretching parameter, the claimed jump relation follows from standard scattering theory.

To extend the result to complex stretching parameters, let $x \in \partial D(\rho)$ and $\{x_n\} \subset D(\rho)$ be any sequence converging toward x. By denoting

$$f_n(s) = \int_{\partial D(\rho)} \partial_{c(y)} \tilde{\Phi}_s^0(x_n,y)dS(y),$$

we observe that Lemma 5.2 implies the claim if we can prove that $\{f_n\}$ is a normal family in a domain $A \subset \mathbb{C}$ containing a segment S of the positive real axis. We choose here

$$A = \{z \in \mathbb{C} \mid 0 < |z| < 2, \ -\delta < \arg z < \frac{\pi}{2} + \delta\},$$

where $\delta > 0$ is small. To show the analyticity, note first that for $x_n \in D(\rho)$ fixed, we have $|\partial_{c(y)} \tilde{\Phi}_s^0(x_n,y)| \leq C\mathrm{dist}\,(x_n, \partial D(\rho))^{-n+1} \leq M_n$, so by Lemma 5.3, f_n is analytic. To see the uniform boundedness, we apply Lemma 5.1 with $k = 0$, $u = 1$ to get

$$f_n(s) = \int_{\partial D(\rho)} b(y)\tilde{\Phi}_s^0(x_n,y)dS(y) - 1.$$

Here, the integral on the right is uniformly bounded with respect to x_n, $s \in A$. Since $\{f_n(s)\}$ converges on the real line, we obtain the desired extension to A. □

To finish this section, we construct the fundamental solution of the perturbed Bérenger operator $\tilde{\Delta} + A - k^2$ constructed in the proof of Lemma 2.3.

Lemma 5.4 *Let A be the operator defined in Lemma 2.3. When $\|A\|$ is small enough, there is a unique solution $\tilde{\Phi}_{(A)}$ of the Lippmann-Schwinger equation*

$$\tilde{\Phi}_{(A)}(x,y) - \int_{D_1} \tilde{\Phi}(x,z) A \tilde{\Phi}_{(A)}(z,y) dz = \tilde{\Phi}(x,y) \qquad (48)$$

where A operates on $z \mapsto \tilde{\Phi}_{(A)}(z,y)$. This solution satisfies

$$(\tilde{\Delta} + A + k^2)\tilde{\Phi}_{(A)}(x,y) = -\delta(x-y) \text{ in } \mathbb{R}^n, \qquad (49)$$

as well as the same decay estimates (28) as $\tilde{\Phi}$.

Proof. Denote by $S : L^2(D_1) \to L^2(D_1)$ the operator

$$Su(x) = -\int_{D_1} \tilde{\Phi}(x,z) A u(z) dz.$$

When the norm of A is small enough, the norm of S is less than one and thus the equation

$$(I + S)u_y = \tilde{\Phi}(\,\cdot\,, y)$$

in $L^2(B_1)$ has a unique solution. This solution can be extended to the whole space by substituting u_y in the Lippmann-Schwinger equation (48), i.e., we define

$$\tilde{\Phi}_{(A)}(x,y) = \tilde{\Phi}(x,y) + \int_{D_1} \tilde{\Phi}(x,z) A u_y(z) dz, \qquad (50)$$

which is the unique solution of (48). The uniqueness is seen by observing that the operator S is compact and using a Fredholm alternative argument. Obviously it satisfies the equation (49).

The asymptotic behaviour of this solution follows from the asymptotics of $\tilde{\Phi}$, since the set D_1 is bounded. □

6 Construction of the solutions

The more or less technical details are applied in this section to prove the existence, uniqueness of the Bérenger solutions in different domains. These in turn are used to construct the double surface operators that play a crucial role in our argument.

We start with the most fundamental result, the proof of Theorem 2.2.

Proof of Theorem 2.2: The existence of a Bérenger solutions of the system (17)–(18) follows from the very construction of the Bérenger equation. It remains therefore to prove the uniqueness. Let v satisfy the equations (17)–(18) with $g = 0$. By Lemma 5.1, we then have

$$v(x) = \int_{\partial\Omega} \frac{\partial\tilde{\Phi}}{\partial n(y)}(x,y)v(y)dS(y), \quad x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

Denote by w the function

$$w(x) = \int_{\partial\Omega} \frac{\partial\Phi}{\partial n(y)}(x,y)v(y)dS(y), \quad x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

This function satisfies the Helmholtz equation and the Sommerfeld radiation condition at infinity. Moreover, $w|_{D\setminus\overline{\Omega}} = v|_{D\setminus\overline{\Omega}}$, so $\partial w/\partial n|_{\partial\Omega} = 0$. By the uniqueness of the scattering solution, we therefore have $w = 0$ and especially, $v|_{D\setminus\overline{\Omega}} = 0$. But by Lemma 4.2, $\tilde{\Delta}$ is elliptic, so the Unique Continuation Principle of elliptic equations ensures that $v = 0$ everywhere. □

Next we prove Lemma 2.4 with the assumption that ρ_1 is in the exponential range.

Proof of Lemma 2.4: We start by proving the uniqueness of the exterior Dirichlet solution. Therefore, assume that $u \in H^1_{(k-\varepsilon)}(\mathbb{R}^n \setminus \overline{D}_1)$ satisfies (30) –(31) with $f = 0$. By Lemma 4.3, we have the estimate

$$\|u\|^2_{H^1(D(\rho)\setminus\overline{D}_1)} \leq \frac{1}{\cos(n\delta)} \max\left(\frac{1}{k^2}, \sup|b_j|\right) \int_{\partial D(\rho)} |n^{\mathrm{T}} H(\operatorname{grad} u)\overline{u}j|dS,$$

the supremum being taken over values in $D(\rho) \setminus \overline{D}_1$. The claim follows from the growth condition (6) and the definition of $H^1_{(k-\varepsilon)}(\mathbb{R}^n \setminus \overline{D}_1)$.

To show the existence, we note that since $x \mapsto \tilde{\Phi}_{(A)}(x,y)$ with its derivatives has exponentially vanishing asymptotics, the function $u = \tilde{K}_{(A),\partial B_1,\mathbb{R}^n\setminus\overline{D}_1}\varphi$ satisfies the Bérenger equation in the exterior domain and it vanishes also exponentially. If φ satisfies (34) it follows from Theorem 5.1 that u has the right boundary condition. Hence it is enough to show that the equation (34) is uniquely solvable. By Fredholm alternative, it is enough to show that the solution is unique. Hence, assume that $\mathrm{Ker}\,(\frac{1}{2}+\tilde{K}_{(A),\partial D_1})\neq\{0\}$ which is equivalent to $\mathrm{Ker}\,(\frac{1}{2}+\tilde{K}^{\mathrm{T}}_{(A),\partial D_1})\neq\{0\}$. Thus, assume that ψ satisfies

$$(\frac{1}{2}+\tilde{K}^{\mathrm{T}}_{(A),\partial D_1})\psi=0. \tag{51}$$

Define now

$$v(x)=\begin{cases}\tilde{S}^{\mathrm{T}}_{(A),\partial D_1,D_1}\psi(x), & x\in D_1,\\ \tilde{S}^{\mathrm{T}}_{(A)\partial D_1,\mathbb{R}^n\setminus\overline{D}_1}\psi(x), & x\in\mathbb{R}^n\setminus\overline{D}_1.\end{cases}$$

Since $\partial_c\tilde{S}^{\mathrm{T}}_{(A),\partial D_1,D_1}\psi|_{\partial D_1}=(1/2+\tilde{K}^{\mathrm{T}}_{(A),\partial D_1})\psi$, the equation (51) implies that v satisfies

$$\begin{aligned}(\tilde{\Delta}^{\mathrm{T}}+k^2+A^{\mathrm{T}})v &= 0 \text{ in } D_1,\\ \partial_c v|_{\partial D_1} &= 0\end{aligned}$$

and by Lemma 2.3, we have $v=0$ in D_1. Since v is a continuous function across ∂D_1 and $A^{\mathrm{T}}=0$ outside D_1, v satisfies

$$\begin{aligned}(\tilde{\Delta}^{\mathrm{T}}+k^2)v &= 0 \text{ in } \mathbb{R}^n\setminus\overline{D}_1\\ v|_{\partial D_1} &= 0\end{aligned}$$

and by definition of v as a layer potential, it decays exponentially at infinity. Let us define $u(x)=J(x)^{-1}v(x)$ in $\mathbb{R}^n\setminus\overline{D}_1$. Since by formula (46) we have $(\tilde{\Delta}^{\mathrm{T}}+k^2)=J(\tilde{\Delta}+k^2)J^{-1}$, the function u solves the Bérenger equation in the exterior domain with a homogenous Dirichlet condition on ∂D_1. Thus, we conclude that $u=0$ and so $v=0$ in $\mathbb{R}^n\setminus D_1$. By using Lemma 5.1, we obtain $\psi=\partial_c v|^+_{\partial D_1}-\partial_c v|^-_{\partial D_1}=0-0=0$. This proves the assertion. $\square$

As a consequence, we obtain the proof of Theorem 2.4.

Proof of Theorem 2.4: The Bérenger solution u_{B} restricted to $\mathbb{R}^n\setminus\overline{D}_1$ is in $H^1_{(k-\varepsilon)}(\mathbb{R}^n\setminus\overline{D}_1)$ and satisfies (30)–(31) with $f=u_{\mathrm{B}}|_{\partial D_1}$, so by Lemma 2.4, we have

$$\begin{aligned}u_{\mathrm{B}}|_{\partial D_2} &= \tilde{K}_{(A),\partial D_1,\partial D_2}(\frac{1}{2}+\tilde{K}_{(A),\partial D_1})^{-1}u_{\mathrm{B}}|_{\partial D_1}\\ &= P_{\mathrm{B}}(u_{\mathrm{B}}|_{\partial D_1}).\end{aligned}$$

To show the uniqueness, assume that $u \in H^1(D_2 \setminus \overline{\Omega})$ is a solution of the system. Define a function v in the exterior domain $\mathbb{R}^2 \setminus \overline{D}_1$ by the equation

$$v = \tilde{K}_{(A),\partial D_1,\mathbb{R}^2\setminus\overline{D}_1}(\frac{1}{2} + \tilde{K}_{(A),\partial D_1})^{-1}(u|_{\partial D_1}).$$

Then $v|_{\partial D_j} = u|_{\partial D_j}$, $j = 1, 2$. This implies by Lemma 4.3 that $v|_{D_2\setminus\overline{D}_1} = u|_{D_2\setminus\overline{D}_1}$.

Let w be given by

$$w(x) = \begin{cases} u(x), & x \in D_2 \setminus \overline{\Omega}, \\ v(x), & x \in \mathbb{R}^2 \setminus \overline{D}_2. \end{cases}$$

Evidently, w vanishes exponentially at infinity and satisfies the Bérenger equation with the boundary condition $\partial w/\partial n|_{\partial\Omega} = g$. By the uniqueness of the solution of the Bérenger system, we must have $w = u_{\mathrm{B}}$. Especially $u = u_{\mathrm{B}}|_{D_2\setminus\overline{\Omega}}$. □

Finally, we need to show that there is an approximating double surface operator that corresponds to the truncated Bérenger problem (19)–(21). As a first step we prove the following auxiliary result.

Lemma 6.1 *Let ρ_1 be in the exponential range and $\rho > \rho_1$. The problem*

$$(\tilde{\Delta} + k^2)v = 0 \text{ in } D(\rho) \setminus \overline{D}_1, \tag{52}$$

$$v|_{\partial D_1} = 0, \tag{53}$$

$$v|_{\partial D(\rho)} = \tilde{f} \in H^{1/2}(\partial D(\rho)) \tag{54}$$

has a unique solution in $H^1(D(\rho) \setminus \overline{D}_1)$. Let Π_ρ denote the mapping

$$\Pi_\rho : H^{1/2}(\partial D(\rho)) \to H^1(D(\rho) \setminus \overline{D}_1), \quad \tilde{f} \mapsto u.$$

The norm of this mapping satisfies the growth estimate

$$\lim_{\rho\to\infty} e^{-\varepsilon\tau(\rho)} \|\Pi_\rho\| = 0$$

for all $\varepsilon > 0$.

Proof: The uniqueness follows immediately from Lemma 4.3. For the existence and norm estimate, let $V : H^{1/2}(\partial D(\rho)) \to H^1(D(\rho) \setminus \overline{D}_1)$ be a right inverse of the trace mapping to the boundary $\partial D(\rho)$ with the property that for $f \in H^{1/2}(\partial D(\rho))$, $V(f)|_{\partial D_1} = 0$. It is not hard to see, by a scaling argument, that we may choose the operator V so that it satisfies a norm estimate

$$\|V(f)\|_{H^1(D(\rho)\setminus\overline{D}_1)} \leq C\rho^{1/2}\|f\|_{H^{1/2}(\partial D(\rho))}.$$

We seek to solve the system (52)–(54) in the form

$$u = V(\tilde{f}) + u_0,$$

where u_0 satisfies

$$\begin{aligned} (\tilde{\Delta} + k^2)u_0 &= -(\tilde{\Delta} + k^2)V(\tilde{f}) \in H^{-1}(D(\rho) \setminus \overline{D}_1), \\ u_0|_{\partial D_1} &= u_0|_{\partial D(\rho)} = 0. \end{aligned}$$

Let L denote the operator

$$L : H_0^1(D(\rho) \setminus \overline{D}_1) \to H^{-1}(D(\rho) \setminus \overline{D}_1), \quad Lu = (\tilde{\Delta} + k^2)u,$$

where $H_0^1(D(\rho) \setminus \overline{D}_1)$ is the Sobolev space of functions with vanishing traces at the boundaries. By Lemma 4.3, we obtain the bound

$$\|\varphi\|_{H_0^1(D(\rho)\setminus\overline{D}_1)} \leq \frac{1}{\cos(n\delta)} \max(k^{-2}, \sup|b_j(x)|)\|L\varphi\|_{H^{-1}(D(\rho)\setminus\overline{D}_1)} \tag{55}$$

implying that L is injective. To show that L has a dense range, assume that for some $u_0 \in H_0^1(D(\rho) \setminus \overline{D}_1)$,

$$\langle u_0, Lv\rangle = 0 \text{ for all } v \in H_0^1(D(\rho) \setminus \overline{D}_1).$$

Here, $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality between $H_0^1(D(\rho)\setminus\overline{D}_1)$ and $H^{-1}(D(\rho)\setminus\overline{D}_1)$. It follows that

$$(\tilde{\Delta} + k^2)^{\mathrm{T}}u_0 = 0.$$

By denoting $v_0(x) = J(x)^{-1}u_0(x)$ and using the formula (46),

$$(\tilde{\Delta} + k^2)^{\mathrm{T}} = J(x)(\tilde{\Delta} + k^2)J(x)^{-1},$$

we observe that $Lv_0 = 0$, implying that $v_0 = u_0 = 0$. Thus, L is invertible, and we have the solution u of the problem (52)–(54) as

$$u = (1 - L^{-1}(\tilde{\Delta} + k^2))V(\tilde{f}).$$

The asymptotic norm estimate follows now from (55), definition of $\tilde{\Delta}$ and the growth restrictions on τ' and τ''. $\Box$

As a corollary, we get Theorem 2.5.

Proof of Theorem 2.5: Consider the Dirichlet problem

$$(\tilde{\Delta}+k^2)u = 0 \text{ in } D(\rho)\setminus\overline{D}_1, \tag{56}$$

$$u|_{\partial D_1} = f\in H^{1/2}(\partial D_1), \tag{57}$$

$$u|_{\partial D(\rho)} = 0. \tag{58}$$

This problem has at most one solution by Lemma 4.3, and in fact the solution can be constructed in terms of previously defined ones as

$$u = v + w,$$

where v satisfies the exterior Dirichlet problem (30)–(31) and w is the solution of (52)–(54) with $\tilde{f} = -v|_{\partial D(\rho)}$. In operator notation, we have

$$\begin{aligned} u \;=\; & \tilde{K}_{(A),\partial D_1, D(\rho)\setminus\overline{D}_1}(\frac{1}{2}+\tilde{K}_{(A),\partial D_1})^{-1}f \\ & -\Pi_\rho\tilde{K}_{(A),\partial D_1,\partial D(\rho)}(\frac{1}{2}+\tilde{K}_{(A),\partial D_1})^{-1}f. \end{aligned}$$

Comparing this representation to the definition of P_{B} we obtain

$$u|_{\partial D_2} = P_{\mathrm{B}}f-\delta P_\rho f = P_\rho f,$$

where δP_ρ is the perturbation from the full space Bérenger double surface operator,

$$\delta P_\rho = \mathrm{Tr}_{\partial D_2}\Pi_\rho\tilde{K}_{(A),\partial D_1,\partial D(\rho)}(\frac{1}{2}+\tilde{K}_{(A),\partial D_1})^{-1}.$$

Here, $\mathrm{Tr}_{\partial D_2}$ denotes the trace mapping $H^1(D_2\setminus\overline{D}_1)\to H^{1/2}(\partial D_2)$. To obtain the norm estimate for $P_{\mathrm{B}}-P_\rho$, we show that

$$\lim_{\rho\to\infty} e^{(k-\varepsilon)\tau(\rho)}\|\tilde{K}_{(A),\partial D_1,\partial D(\rho)}\| = 0,$$

the norm being the operator norm $H^{1/2}(\partial D_1)\to H^{1/2}(\partial D(\rho))$. Clearly it is sufficient to show the above limit claim when the operator is interpreted as

$$\tilde{K}_{(A),\partial D_1,\partial D(\rho)} : L^2(\partial D_1)\to C^1(\partial D(\rho)).$$

But this norm estimate follows directly from the asymptotic behavior of the Green's function $\tilde{\Phi}_{(A)}$ and its derivatives.

To complete the proof, we need to discuss the equivalence of the truncated Bérenger problem and the double surface operator system. If u satisfies the truncated problem, it satisfies the Dirichlet problem (56)–(58) with $f = u|_{\partial D_1}$ and therefore the double surface condition. Conversely, if u satisfies the double surface operator system, we easily see that u is

$$\begin{aligned} u &= \tilde{K}_{(A),\partial D_1, D(\rho)\setminus\overline{D}_1}(\frac{1}{2} + \tilde{K}_{(A),\partial D_1})^{-1}(u|_{\partial D_1}) \\ &\qquad - \Pi_\rho \tilde{K}_{(A),\partial D_1,\partial D(\rho)}(\frac{1}{2} + \tilde{K}_{(A),\partial D_1})^{-1}(u|_{\partial D_1}) \end{aligned}$$

everywhere in $D(\rho) \setminus \overline{D}_1$ and so $u|_{\partial D(\rho)} = 0$. □

7 Appendix: Representation of the Bérenger equation in tangent–normal coordinates

We consider the Bérenger equation using the coordinate system generated by the parameterization of the surface ∂D. Assume that ∂D has a local parameterization of the form

$$p = p(u) \in \partial D, \quad u = (u^1, \dots, u^{n-1}) \in \mathbb{R}^{n-1}.$$

This parameterization yield immediately a tangent–normal coordinate system for the exterior domain $\mathbb{R}^n \setminus D$,

$$x(u; h) = p(u) + hn(u),$$

where $h = \mathrm{dist}(x, \partial D) = |x - p(u)|$ as before and $n(u)$ is the exterior unit normal at $p(u)$. In order to calculate the fundamental matrix of the exterior domain in these coordinates, let us denote by $g(u) = (g(u)_{i,j}) \in \mathbb{R}^{(n-1)\times(n-1)}$ the fundamental matrix of ∂D in u–coordinates,

$$g_{i,j} = \left(\frac{\partial p}{\partial u^i}\right)^{\mathrm{T}} \left(\frac{\partial p}{\partial u^j}\right), \quad 1 \le i, j \le n-1.$$

Further, denote by $B(u) = (b_\ell^k(u))_{1\le k,\ell\le n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$ the matrix whose elements are determined by the equation

$$\frac{\partial n}{\partial u^\ell}(u) = \sum_{k=1}^{n-1} b_\ell^k(u) \frac{\partial p}{\partial u^k}(u).$$

For $1 \le i \le n-1$, we have

$$\frac{\partial x}{\partial u^i} = \sum_{k=1}^{n-1} (\delta_i^k + h b_i^k) \frac{\partial p}{\partial u^k},$$

and a straightforward calculation then shows that for $1 \le i, j \le n-1$,

$$\begin{aligned} G_{i,j} &= \left(\frac{\partial x}{\partial u^i}\right)^{\mathrm{T}} \left(\frac{\partial x}{\partial u^j}\right) \\ &= g_{i,j} + h \sum_{k=1}^{n-1} (g_{i,k} b_j^k + b_i^k g_{k,j}) + h^2 \sum_{k,\ell=1}^{n-1} b_i^k g_{k,\ell} b_j^\ell \\ &= (g + h(gB + B^{\mathrm{T}} g) + h^2 B^{\mathrm{T}} g B)_{i,j}. \end{aligned}$$

Using the formula $g^{-1} B^{\mathrm{T}} g = B$, or in component form

$$\sum_{k,\ell=1}^{n-1} g^{i,k} b_k^\ell g_{\ell,j} = b_j^i,$$

we obtain

$$g + h(gB + B^{\mathrm{T}} g) + h^2 B^{\mathrm{T}} g B = g\gamma,$$

where where $\gamma(u;h) = (\gamma_\ell^k(u;h)) \in \mathbb{R}^{(n-1)\times(n-1)}$ is

$$\gamma = I + 2hB + h^2 B^2 = (I + hB)^2.$$

Therefore, we have

$$G(u;h) = \begin{pmatrix} g(u) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma(u;h) & 0 \\ 0 & 1 \end{pmatrix} = G_0(u) C(u;h).$$

To find a formal complexification of the fundamental matrix corresponding to the stretched coordinates, we write the complexified variable $\tilde{x} = F(x)$ as

$$\tilde{x}(u;h) = p(u) + (h + i\tau(h)) n(u).$$

Thus, the complexified fundamental tensor $\tilde{G}(u;h)$ attains the form

$$\tilde{G}(u;h) = \begin{pmatrix} g(u) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma(u;h+i\tau(h)) & 0 \\ 0 & (1+i\tau'(h))^2 \end{pmatrix} = G_0(u)\tilde{C}(u;h).$$

Since the Laplacian in the coordinate system $(u;h)$ reads as

$$\Delta v = \frac{1}{\sqrt{|G|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial u^i} \left(\sqrt{|G|} G^{i,j} \frac{\partial v}{\partial u^j} \right),$$

where $u_n = h$, $(G^{i,j}) = (G_{i,j})^{-1}$ and $|G| = \mathrm{Det}\, G$, a formal complexification yields the operator

$$\Delta^* v = \frac{1}{\sqrt{|\tilde{G}|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial u^i} \left(\sqrt{|\tilde{G}|} \tilde{G}^{i,j} \frac{\partial v}{\partial u^j} \right).$$

We will show that in this coordinate system, we have $\Delta^* = \tilde{\Delta}$.

Lemma 7.1 *The Bérenger equation can be witten in the tangent–normal coordinate system as*

$$(\Delta^* + k^2)u = 0.$$

In particular, this equation allows a divergence form representation

$$\mathrm{div}\,(A\,\mathrm{grad} v) + k^2 \beta v = 0,$$

where

$$\beta = \beta(u,h) = \left(\frac{|\tilde{C}(u;h)|}{|C(u;h)|} \right)^{1/2},$$

and

$$A = \beta(u;h) \begin{pmatrix} \gamma(u;h+i\tau(h))^{-1}\gamma(u;h) & 0 \\ 0 & (1+i\tau'(h))^{-2} \end{pmatrix}.$$

Proof: For $s > 0$, the transformation $F_s : \mathbb{R}^n \to \mathbb{R}^n$ is a C^2–diffeomorphism, and the fundamental tensor transforms as

$$G(u;h) \to G_s(u;h) = \begin{pmatrix} g(u) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma(u;h+s\tau(h)) & 0 \\ 0 & (1+s\tau'(h))^2 \end{pmatrix}.$$

Consequently, the Laplacian in these new coordinates becomes

$$\Delta_s^* v = \frac{1}{\sqrt{|G_s|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial u^i} \left(\sqrt{|G_s|} G_s^{i,j} \frac{\partial v}{\partial u^j} \right),$$

and by definition, $\Delta_s^* = \tilde{\Delta}_s$. By writing the operators $\tilde{\Delta}_s$ and $\hat{\Delta}_s$ in canonical form, we see that the coefficients of these operators are analytic functions of s in the neighborhood of the set $\mathrm{Re} s \geq 0$. and so they are identical in the whole domain. Since

$$\mathrm{div}\,(A \mathrm{grad}\, u) = \beta \Delta^*,$$

we see by looking the second order coefficients that $A = \beta H^{\mathrm{T}} H$. □

From the divergence form, we can easily obtain the equation e.g. in the spherical coordinates as given e.g. in [5].

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