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Abstract: *We consider the asymptotic consistency error associated to a modified bilinear finite element approximation of a simple parameter dependent elliptic problem, the problem of anisotropic heat conduction. We prove that when approximating smooth solutions with bilinear finite elements the consistency error due to the variational crime is negligible compared to the approximation error, whereas for nonsmooth solutions the performance of the finite element scheme, as shown by a numerical example, depends on the mesh.*

AMS subject classifications: 65N30, 73K15

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1 Introduction

In several parameter-dependent problems we encounter a constrained problem when the associated parameter ϵ approaches a limit value ϵ_0. In the limit case the variational problem is posed in a subspace $\mathcal{V}_0$ of the energy space $\mathcal{V}$ so as to take into account the constraints imposed by the special parameter value. Well-know examples include the problems of bending a thin plate or shell [4, 5] and the case of an anisotropic heat conduction [1, 3].

Usually the intersection of the constrained solution space $\mathcal{V}_0$ with the finite-element space $\mathcal{V}_h$ is very small and sometimes even $\mathcal{V}_0 \cap \mathcal{V}_h = \{0\}$. This leads to a drastic loss in the approximation properties of low-order finite-element schemes for parameter values near the limit value. In order to avoid this undesirable phenomenon known as locking, and to retain the approximation properties of the finite-element scheme, a modified mesh dependent variational formulation is often introduced. The approximate solution is then sought in a somewhat different space $\mathcal{V}_{0,h} \subset \mathcal{V}_h$ which is aimed to be larger than $\mathcal{V}_0 \cap \mathcal{V}_h$. However, it is then necessary that $\mathcal{V}_{0,h} \not\subset \mathcal{V}_0$ and so this modification gives rise to a consistency error which depends on the structure of $\mathcal{V}_{0,h}$. It is often easily justified that the consistency error component – called the asymptotic consistency error – is negligible. This is especially true when the solution u_0 is smooth. On the other hand, for non-smooth solutions this error component can have significant importance when one is trying to resolve the problem with a parameter value near to the limit value ϵ_0.

In this paper we briefly examine the asymptotic consistency error in view of a model problem already discussed in [3]. We show that when the asymptotic solution u_0 is smooth the asymptotic consistency error is $O(h)$ when using bilinear elements. In case of a nonsmooth u_0 we show by an example that the error can be $O(h^{1/2})$ or even $O(h^{1/3})$ depending on the mesh.

The plan of the paper is as follows. In Section 2 we present our model problem, and the modified variational formulation is introduced in Section 3. Section 4 is concerned with the asymptotic consistency error for smooth solutions and the nonsmooth case is treated in Section 5.

The kth Sobolev norm over the assumed domain (unit square) is denoted by $|| \cdot ||_k$ and the corresponding seminorm by $| \cdot |_k$. Other domains will be denoted by an additional subscript. The L^2-inner product is written as $< \cdot, \cdot >$. Finally, C denotes a constant, not necessarily always the same, but independent of any parameters unless explicitly stated otherwise.

2 The model problem

As our model problem we take the problem of anisotropic heat conduction in the unit square $\Omega = (0,1) \times (0,1)$ as in [1, 3]:

$$-\frac{\partial^2 u}{\partial \xi^2} - \epsilon^2 \frac{\partial^2 u}{\partial \eta^2} = f \text{ in } \Omega$$

where

$$\begin{cases} \xi &= \alpha x + \beta y \\ \eta &= -\beta x + \alpha y \end{cases}$$

with $\alpha^2 + \beta^2 = 1$, $\alpha, \beta \neq 0$. To this equation we associate three different boundary conditions:

A. $u = w$ on $\partial\Omega$

B. $\partial_\nu u = g$ when $x = 1$ or $y = 1$ and $u = w$ elsewhere on $\partial\Omega$

C. $\partial_\nu u = g$ on $\partial\Omega$

Here f, w and g are given functions or suitably chosen distributions and ∂_ν stands for the normal component of the heat flux at the boundary, i.e.,

$$\partial_\nu u = (\alpha n_x + \beta n_y)\frac{\partial u}{\partial \xi} + \epsilon^2(-\beta n_x + \alpha n_y)\frac{\partial u}{\partial \eta},$$

where (n_x, n_y) is the outward unit normal to $\partial\Omega$. The variational formulation of these problems is: Find $u \in \mathcal{V}$ such that

$$\mathcal{A}(u, v) = \phi(v) \quad \forall v \in \mathcal{V}^0 \tag{2.1}$$

where the bilinear form is given by

$$\mathcal{A}(u, v) =< \frac{\partial u}{\partial \xi}, \frac{\partial v}{\partial \xi} > +\epsilon^2 < \frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \eta} >,$$

and the linear functional as

$$\phi(v) =< f, v > + \int_{\Gamma_N} gv d\Gamma,$$

where $\Gamma_N = \emptyset$ for Problem A, $\Gamma_N = \{(x, y) \in \partial\Omega \,|\, x = 1 \text{ or } y = 1\}$ for Problem B, and $\Gamma_N = \partial\Omega$ for Problem C. In (2.1), $\mathcal{V} = H^1(\Omega)$ and $\mathcal{V}^0 = \{v \in H^1(\Omega) \,|\, v = 0 \text{ on } \partial\Omega \backslash \Gamma_N\}$. In Problem C we assume that $\phi(1) = 0$ and impose the constraint $< u, 1 >= 0$ on $\mathcal{V}$ so as to make the solution unique.

Depending on the Problem, the solution can fall into two different asymptotical states when $\epsilon \to 0$. In Problems A and B we may assume that

$$u \sim u^0 \text{ as } \epsilon \to 0$$

where the limiting solution u^0 satisfies

$$< \frac{\partial u^0}{\partial \xi}, \frac{\partial v}{\partial \xi} >= \phi(v) \quad \forall v \in \mathcal{V}^0. \tag{2.2}$$

so that this corresponds to the "cool" state [3].

In Problem C the solution can have a "hot" component so that

$$u \sim \epsilon^{-2} u^0 \text{ as } \epsilon \to 0.$$

Here u^0 (the scaled limiting solution) is given by: Find $u_0 \in \mathcal{V}_0$ such that

$$\mathcal{A}(u_0, v) =< \frac{\partial u^0}{\partial \eta}, \frac{\partial v}{\partial \eta} >= \phi(v) \quad \forall v \in \mathcal{V}_0 \tag{2.3}$$

where $\mathcal{V}_0 = \{v \in \mathcal{V} \mid \frac{\partial v}{\partial \xi} = 0\}$.

3 The reduced-flux formulation

In the finite element scheme to be studied we assume that Ω is subdivided into convex disjoint quadrilaterals K satisfying the usual shape regularity assumptions (cf. [2]). We denote by h_K the largest side of an element K and write $h = \max_K h_K$. On this mesh we denote the piecewise bilinear finite element space by $\mathcal{V}_h$. It was shown in [3] that the conventional approach based on (2.1) suffers from severe locking in Problem C. In order to improve the approximation properties in the "hot" state a new mesh-dependent formulation can be introduced: Find $u_h \in \mathcal{V}_h$ such that

$$\mathcal{A}_h(u_h, v) = \phi(v) \quad \forall v \in \mathcal{V}_h^0 \tag{3.1}$$

where

$$\begin{aligned} \mathcal{A}_h(u,v) = &< R_h \frac{\partial u}{\partial \xi}, R_h \frac{\partial v}{\partial \xi} > + \epsilon^2 < \frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \eta} > \\ &+ \epsilon^2 < (I - R_h) \frac{\partial u}{\partial \xi}, \frac{\partial v}{\partial \xi} > \end{aligned} \tag{3.2}$$

and $\mathcal{V}_h^0$ stands for the subspace of $\mathcal{V}_h$ with the same constraints as in the exact formulation. Further, R_h is a numerical flux-reduction operator already discussed in [3] chosen to be the orthogonal L^2-projection onto elementwise constant functions, i.e.

$$(R_h \varphi)(x,y) = \frac{1}{\text{area}(K)} \int_K \varphi(x', y')\, dx' dy', \quad (x,y) \in K$$

for every element K inducing the space $\mathcal{V}_{0,h} \subset \mathcal{V}_h$ defined by

$$\mathcal{V}_{0,h} = \{ v \in \mathcal{V}_h \mid R_h v = 0 \}.$$

The modified formulation (3.2) was analyzed in [3] and it was shown that for sufficiently smooth solutions and regular meshes both the approximation and the consistency error have optimal bounds in the modified energy norm $|||\cdot|||_h = \sqrt{\mathcal{A}_h(\cdot,\cdot)}$ when $\epsilon > 0$.

On the other hand, in Problem C when $\epsilon = 0$ we have that

$$< \frac{\partial u_0}{\partial \eta}, \frac{\partial v}{\partial \eta} > = \phi(v) \quad \forall v \in \mathcal{V}_0$$

and

$$< \frac{\partial u_h}{\partial \eta}, \frac{\partial v}{\partial \eta} > + < \frac{\partial u_h}{\partial \xi}, \frac{\partial v}{\partial \xi} > = \phi(v) \quad \forall v \in \mathcal{V}_{0,h} \tag{3.3}$$

for the corresponding (scaled) finite-element solution u_h. Let $\tilde{u}_h$ be the best finite-element approximation to u_0 in $\mathcal{V}_{0,h}$, i.e.

$$< \frac{\partial \tilde{u}_h}{\partial \eta}, \frac{\partial v}{\partial \eta} > + < \frac{\partial \tilde{u}_h}{\partial \xi}, \frac{\partial v}{\partial \xi} > = < \frac{\partial u_0}{\partial \eta}, \frac{\partial v}{\partial \eta} > \quad \forall v \in \mathcal{V}_{0,h} \tag{3.4}$$

so that by (3.3), (3.4), and denoting $z_h = u_h - \tilde{u}_h$

$$
\begin{aligned}
&< \frac{\partial z_h}{\partial \eta}, \frac{\partial v}{\partial \eta} > + < \frac{\partial z_h}{\partial \xi}, \frac{\partial v}{\partial \xi} > \\
&= \phi(v) - < \frac{\partial u_0}{\partial \eta}, \frac{\partial v}{\partial \eta} > \quad \forall v \in \mathcal{V}_{0,h}.
\end{aligned} \tag{3.5}
$$

Thus, the asymptotic consistency error is given by

$$
e_C^0 = |||z_h|||_{0,h} = \sup_{v \in \mathcal{V}_{0,h}, v \neq 0} \frac{\phi(v) - < \frac{\partial u_0}{\partial \eta}, \frac{\partial v}{\partial \eta} >}{|||v|||_{0,h}} \tag{3.6}
$$

where $|||\cdot|||_{0,h} = |\cdot|_1$ is the modified energy norm in $\mathcal{V}_{0,h}$ at $\epsilon = 0$. We note that $z_h \in \mathcal{V}_{0,h}$ can actually be solved from the variational formulation (3.5).

Our aim is to consider the asymptotic consistency error e_C^0 in two different cases with respect to the smoothness of u_0:

(I) The smooth case, when $u_0 \in H^3(\Omega)$ and $\phi(v) = \int_\Omega f v d\xi d\eta$ for some $f \in H^1(\Omega)$.

(II) The non-smooth case when $\phi(v) = v(\frac{3}{4}, \frac{1}{4}) - v(\frac{1}{4}, \frac{3}{4})$ so that $u_0 \in H^{3/2-\sigma}(\Omega)$ for $\sigma > 0$.

In case (I) we show analytically that the asymptotic consistency error is negligible as confirmed by a numerical experiment whereas in case (II) we conduct a similar numerical experiment showing a completely different behavior. In these experiments we assume that $\alpha = \beta = \frac{1}{\sqrt{2}}$ and consider two different meshes:

(S) A mesh consisting of squares with side length $h_x = h_y = h_S$.

(R) A mesh consisting of rectangles with side lengths $h_y = 2h_x = h_R$.

where h_x and h_y are the mesh spacings in x- and y-directions, respectively. We note that $\mathcal{V}_{h_S} \subset \mathcal{V}_{h_R}$ when $h_S = h_R$.

4 The asymptotic consistency error for smooth u_0

Theorem 4.1. *Assume that $u_0 \in H^3(\Omega)$ and $\phi(v) = \int_\Omega f v d\xi d\eta$, $f \in H^1(\Omega)$. Then the asymptotic consistency error defined in (3.6) satisfies*

$$
e_C^0 \leq Ch(||u_0||_3 + ||f||_1).
$$

Proof. Since $<\frac{\partial u_0}{\partial \eta}, \frac{\partial v}{\partial \eta}>= \phi(v) \quad \forall v \in \mathcal{V}_0$ we can write for $\tilde{v} \in \mathcal{V}_{0,h}$

$$
\begin{aligned}
<\frac{\partial u_0}{\partial \eta}, \frac{\partial \tilde{v}}{\partial \eta}> -\phi(\tilde{v}) = &< \frac{\partial u_0}{\partial \eta}, \frac{\partial \tilde{v}}{\partial \eta} - \frac{\partial v}{\partial \eta}> -\phi(\tilde{v}-v) \\
&=- <\frac{\partial^2 u_0}{\partial \eta^2}, \tilde{v}-v> +\alpha < \frac{\partial u_0}{\partial \eta}, \tilde{v}-v>_{\Gamma^1} \\
&\quad + \beta < \frac{\partial u_0}{\partial \eta}, \tilde{v}-v>_{\Gamma^2} +\alpha < \frac{\partial u_0}{\partial \eta}, \tilde{v}-v>_{\Gamma^3} \\
&\quad + \beta < \frac{\partial u_0}{\partial \eta}, \tilde{v}-v>_{\Gamma^4} -\phi(\tilde{v}-v)
\end{aligned}
$$

by integration by parts. Here

$$
\begin{cases}
\Gamma^1 = & \{(x,y) \in \partial\Omega \mid y = 0\} \\
\Gamma^2 = & \{(x,y) \in \partial\Omega \mid x = 0\} \\
\Gamma^3 = & \{(x,y) \in \partial\Omega \mid y = 1\} \\
\Gamma^4 = & \{(x,y) \in \partial\Omega \mid x = 1\}.
\end{cases}
$$

Let us then choose $v \in \mathcal{V}_0$ such that $v = \tilde{v}$ on $\Gamma^1 \cup \Gamma^2$. The constraint $\frac{\partial v}{\partial \xi} = 0$ implies

$$
\begin{aligned}
(\tilde{v}-v)(\xi,\eta) &= \int_{\xi_0(\eta)}^{\xi} \frac{\partial \tilde{v}}{\partial \xi}(\tau,\eta) d\tau \\
&= \int_{\xi_0(\eta)}^{\xi} \frac{\partial \tilde{v}}{\partial \xi}(\tau,\eta) - R_h^{(\tau,\eta)}\big(\frac{\partial \tilde{v}}{\partial \xi}(\tau,\eta)\big) d\tau \qquad (4.1)\\
&= \int_{\xi_0(\eta)}^{\xi} \big(I - R_h^{(\tau,\eta)}\big)\frac{\partial \tilde{v}}{\partial \xi}(\tau,\eta) d\tau
\end{aligned}
$$

by the fact that $R_h \frac{\partial \tilde{v}}{\partial \xi} = 0$. Here $(\xi_0(\eta), \eta) \in \Gamma^1 \cup \Gamma^2$ is the projection of (ξ, η) onto $\Gamma^1 \cup \Gamma^2$ along the ξ-direction. Thus, (4.1) implies that

$$
\begin{aligned}
<\frac{\partial^2 u_0}{\partial \eta^2}, \tilde{v}-v>= &\int_\Omega \int_{\xi_0(\eta)}^{\xi} \frac{\partial^2 u_0}{\partial \eta^2}(\xi,\eta)\big(I - R_h^{(\tau,\eta)}\big)\frac{\partial \tilde{v}}{\partial \xi}(\tau,\eta) d\tau d\xi d\eta \\
= &\int_\Omega \int_{\xi_0(\eta)}^{\xi} \big(I - R_h^{(\tau,\eta)}\big)\frac{\partial^2 u_0}{\partial \eta^2}(\xi,\eta)\frac{\partial \tilde{v}}{\partial \xi}(\tau,\eta) d\tau d\xi d\eta \qquad (4.2)\\
&\le Ch||u_0||_3|\tilde{v}|_1 \le Ch||u_0||_3|||\tilde{v}|||_{0,h}.
\end{aligned}
$$

As for the boundary terms we have that $< \frac{\partial u_0}{\partial \eta}, \tilde{v}-v>_{\Gamma^1}=< \frac{\partial u_0}{\partial \eta}, \tilde{v}-v>_{\Gamma^2}= 0$ and that

$$
\begin{aligned}
<\frac{\partial u_0}{\partial \eta}, \tilde{v}-v>_{\Gamma^3\cup\Gamma^4}= &\int_{\eta_0}^{\eta_1} \int_{\xi_0(\eta)}^{\xi_1(\eta)} \frac{\partial u_0}{\partial \eta}(\xi_1(\eta),\eta)\big(I - R_h^{(\tau,\eta)}\big)\frac{\partial \tilde{v}}{\partial \xi}(\tau,\eta) d\tau d\eta \\
= &\int_{\eta_0}^{\eta_1} \int_{\xi_0(\eta)}^{\xi_1(\eta)} \big(I - R_h^{(\tau,\eta)}\big)\frac{\partial u_0}{\partial \eta}(\xi_1(\eta),\eta)\frac{\partial \tilde{v}}{\partial \xi}(\tau,\eta) d\tau d\eta \qquad (4.3)\\
&\le Ch||u_0||_{2,\Gamma^3\cup\Gamma^4}|\tilde{v}|_1 \le Ch||u_0||_3|||\tilde{v}|||_{0,h}
\end{aligned}
$$

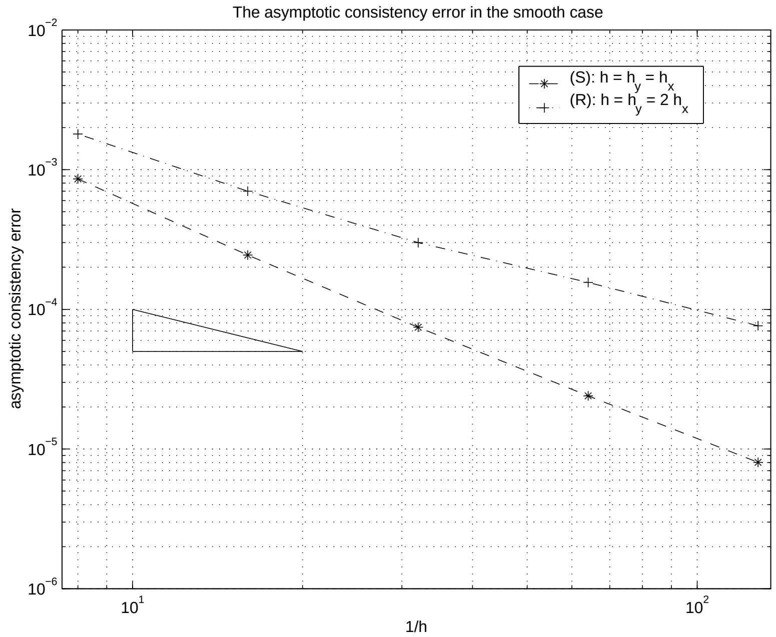

Figure 1: Asymptotic consistency errors e_C^0 in the case of a smooth solution u_0 for meshes (S) and (R) with $\alpha = \beta = \frac{1}{\sqrt{2}}$.

where $(\xi_1(\eta), \eta) \in \Gamma^3 \cup \Gamma^4$ denotes the projection of (ξ, η) onto $\Gamma^3 \cup \Gamma^4$ along the ξ-direction and η_0, η_1 are chosen so that the integration is taken over $\Gamma^3 \cup \Gamma^4$. For the load we have

$$\phi(v - \tilde{v}) = \int_\Omega f(v - \tilde{v}) d\xi d\eta \leq Ch||f||_1|||\tilde{v}|||_{0,h} \tag{4.4}$$

again by (4.1) and by the arguments above. The claim follows from (4.2) – (4.4). $\square$ $\square$

To illustrate the behavior of the asymptotic consistency error in the smooth case we have solved the problem for $\phi(v) = \int_\Omega \sin(2\pi x) v dx dy$ (see also [3]) so that

$$\frac{\partial u_0}{\partial \eta} = \begin{cases} \frac{-2\sqrt{2}\pi\eta + \sin(2\sqrt{2}\pi\eta) + 2\pi}{4(\sqrt{2} - 2\eta)\pi^2} & \text{when } \eta \geq 0 \\ \frac{2\sqrt{2}\pi\eta - \sin(2\sqrt{2}\pi\eta) + 2\pi}{4(\sqrt{2} + 2\eta)\pi^2} & \text{when } \eta < 0. \end{cases}$$

The results obtained solving for z_h in (3.5) for the two different meshes (S) and (R) are shown in Figure 1. We note that the error vanishes rapidly and becomes negligible as can be anticipated from Theorem 4.1.

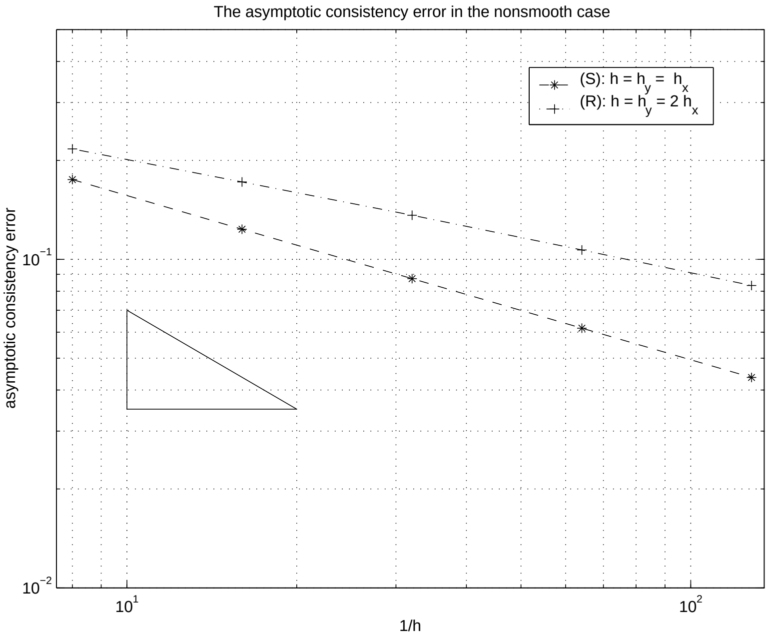

Figure 2: Asymptotic consistency errors e_C^0 in the case of a nonsmooth solution u_0 for meshes (S) and (R) with $\alpha = \beta = \frac{1}{\sqrt{2}}$.

5 The asymptotic consistency error for nonsmooth u_0

To show the behavior of e_C^0 when u_0 is not smooth we consider the case (II). In this case we have

$$\frac{\partial u_0}{\partial \eta} = \begin{cases} \frac{1}{\sqrt{2}-2|\eta|} & \text{when } |\eta| \leq \frac{1}{\sqrt{8}} \\ 0 & \text{when } \frac{1}{\sqrt{8}} < |\eta| \leq \frac{1}{\sqrt{2}}. \end{cases}$$

Again, we solve for z_h in (3.5) and obtain the results shown in Figure 2 for both meshes, (S) and (R). It appears that in both cases $|||z_h|||_{0,h} \to 0$ as $h \to 0$ but in the case of the non-square mesh the convergence is markedly slower. In fact, for the square mesh we have $|||z_h|||_{0,h} \sim O(h^{1/2})$ whereas for non-square meshes we have more like $|||z_h|||_{0,h} \sim O(h^{1/3})$ or slightly better.

To shed some light on this phenomenon we make some remarks on the structure of $\mathcal{V}_{0,h}$ in both cases.

Remark 5.1. In the case of the mesh (S) the condition $R_h \frac{\partial \tilde{v}}{\partial \xi} = 0$ is equivalent to requiring $\tilde{v}^{i-1,j-1} = \tilde{v}^{i,j}$ in terms of the nodal values of $\tilde{v} \in \mathcal{V}_{0,h}$ so that the values at the nodes are preserved on the diagonals $\eta = \text{const}$. Comparing this to the structure of $\mathcal{V}_0$ we see that we can at least find an interpolant $v \in \mathcal{V}_0$ for every $\tilde{v} \in \mathcal{V}_{0,h}$. Also, $\frac{\partial \tilde{v}}{\partial \eta} \in H^{1/2-\sigma}(\Omega)$, $\sigma > 0$, so that the convergence rate for the mesh (S) is as expected. On the other hand, in the case of the mesh (R) the condition $R_h \frac{\partial \tilde{v}}{\partial \xi} = 0$ reads

$$3\left(\tilde{v}^{i,j} - \tilde{v}^{i-1,j-1}\right) = \tilde{v}^{i-1,j} - \tilde{v}^{i,j-1}$$

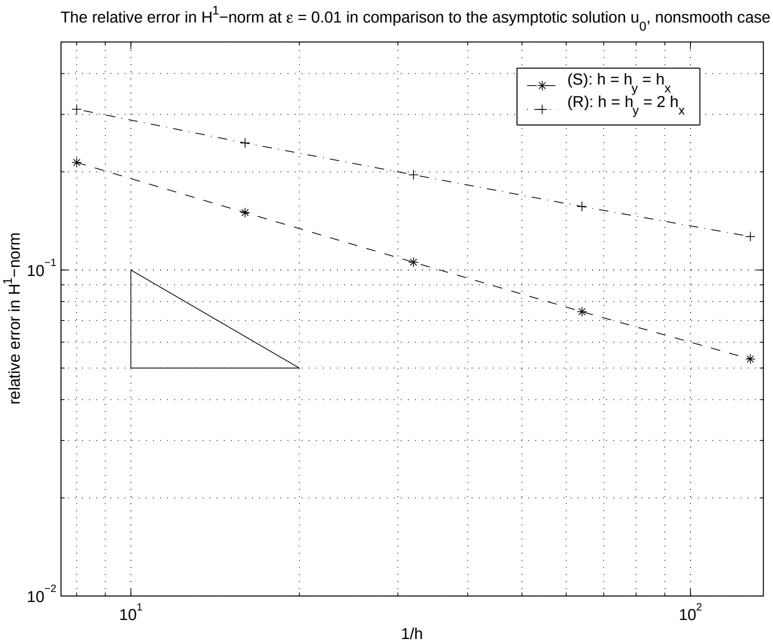

Figure 3: The relative error in the H^1-seminorm for $\epsilon = 0,01$ for meshes (S) and (R) with $\alpha = \beta = \frac{1}{\sqrt{2}}$.

so that the ξ-direction is not preferred in the same way as in the case of the mesh (S).

Remark 5.2. The structure of the local polynomial spaces gives another way of looking at the difference between the cases (S) and (R). In case of the mesh (S) the elementwise expression is given by

$$\tilde{v}_{|K} = c_1 + c_2\eta + c_3\xi + c_4\xi^2 + c_5\eta^2 \tag{5.1}$$

so that locally $\tilde{v}$ lacks the crossterm $\xi\eta$. It follows that $\frac{\partial \tilde{v}}{\partial \eta}_{|K}$ is a function of η only. In the case (R) the expression (5.1) has the additional term $c_6\xi\eta$ so that $\frac{\partial \tilde{v}}{\partial \eta}_{|K}$ depends also on ξ whereas for every $v \in \mathcal{V}_0$ $\frac{\partial v}{\partial \eta}$ is globally a function of η only.

In Figure 3 we present the results obtained with a small, but positive value of ϵ and with the same load as in case (II). Here we have calculated the relative error $|u_h - u_0|_1/|u_0|_1$ in the H^1-seminorm at $\epsilon = 0,01$. It should be noted that the convergence of the relative error is again of the order $O(h^{1/2})$ for the mesh (S) and of the order $O(h^{1/3})$ for the mesh (R). Since the approximation error is only $O(h^{1/2})$ in both cases, the large error component in the case (R) must be due to consistency problems. This is also reflected by the fact that the approximation properties of $\mathcal{V}_{h_R}$ are better than those of $\mathcal{V}_{h_S}$ for $h_S = h_R$, and yet the total error is greater in this case for the mesh (R).

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