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Nikolai Yu. Bakaev

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

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Nikolai Yu. Bakaev

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

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Abstract: *We present some resolvent estimates of elliptic differential and finite element operators in pairs of function spaces, for which the first space in a pair is endowed with stronger norm. In this work we deal with estimates in (Lebesgue, Lebesgue), (Hölder, Lebesgue), and (Hölder, Hölder) pairs of norms. In particular, our results are useful for the stability and error analysis of semi-discrete and fully discrete approximations to parabolic partial differential problems with rough and distribution-valued data.*

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bakaev@postman.ru

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Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/

1 INTRODUCTION

The aim of this work is twofold. The main objective is to get some new resolvent estimates of elliptic finite element operators which are intended for use in applications of the finite element method to parabolic initial-boundary value problems. It however turns out that this might be successfully achieved when using related estimates for elliptic partial differential operators; some of them seem, apart from everything else, to be of independent interest. We therefore start with showing resolvent estimates for differential operators. In both cases, continuous and discrete, we make an emphasis on deriving such estimates in pairs of function spaces, for which the first space in a pair is endowed with stronger norm. More precisely, our consideration deals with estimates in (Lebesgue, Lebesgue), (Hölder, Lebesgue), and (Hölder, Hölder) pairs of norms.

It is well known that resolvent estimates of elliptic partial differential operators are most important for applications of semigroup theory to the analysis of parabolic initial-boundary value problems. In fact such estimates allow one to study the problem of generation of analytic semigroups, associated to the elliptic operators, in different function spaces. The classical papers of Agmon [1] and Stewart [31] are concerned with generation in Lebesgue spaces and in the space of continuous functions, respectively. For recent work concerning generation in Besov and Hölder spaces we refer to Grisward [19], Campanato [8], and Lunardi [21]. The problem of generation in Hölder and Sobolev spaces of integral positive orders was examined in Colombo & Vespri [12] and Mora [23]. A detailed discussion of generation results in different function spaces can be found, e.g., in Lunardi [22].

We further remark that resolvent estimates for elliptic finite element operators are of great significance for applications of operator theory to the analysis of both spatially semi-discrete and fully discrete approximations to parabolic PDE problems (cf., e.g., [7]). For earlier work related to this subject we refer (in the chronological order) to Fujii [18], Schatz, Thomée & Wahlbin [28], Nitsche & Wheleer [24], Wahlbin [36], Thomée & Wahlbin [33], Rannacher [26], Chen [10], Crouzeix, Larsson & Thomée [13], Palencia [25], Bakaev, Larsson & Thomée [6], Schatz, Thomée & Wahlbin [29], Thomée & Wahlbin [34], Bakaev [4], Crouzeix & Thomée [14], and Bakaev, Thomée & Wahlbin [7] (for brief discussions of the above work, see, e.g., [29] and [4]).

Let $\Omega = \text{Int}\ \Omega$ be a convex bounded domain in $\mathbb{R}^d$, $d \geq 1$, with smooth boundary $\partial\Omega$. For simplicity we assume Ω to be of class $\mathcal{C}^\infty$, meaning that in the case $d = 1$, Ω is an open interval of the real axis. We introduce a linear operator A by

$$A = -\Delta = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2},$$

with homogeneous Dirichlet boundary conditions on $\partial\Omega$. The operator A will be considered in complex Lebesgue, Sobolev, and Hölder spaces on Ω (or on $\overline{\Omega}$), for which we use the standard notation: L_p, W_p^l, and $\mathcal{C}^\xi$, respectively. The

respective norms will be denoted by $\|.\|_p$, $\|.\|_{p;l}$, and $|.|_\xi$. In this work it will be convenient to identify denoted differently but, at the same time, equivalent norms, although they are usually intended for use in different underlying spaces. In such a way, we shall often write $|v|_0$ instead of $\|v\|_\infty$ and $|v|_l$ instead of $\|v\|_{l;\infty}$, even for measuring functions v in L_∞ and W^l_∞, respectively. Let further $\dot{\mathcal{C}}^0 = \{v \in \mathcal{C}^0 : v = 0 \text{ on } \partial\Omega\}$, $\dot{\mathcal{C}}^\xi = \mathcal{C}^\xi \cap \dot{\mathcal{C}}^0$, and $\dot{W}^l_\infty = W^l_\infty \cap \dot{\mathcal{C}}^0$. Given two function spaces E_1 and E_2 (among those introduced above), if $B : E_1 \to E_2$ is a linear bounded operator, the symbol $\|\!|B|\!\|_{E_1 \to E_2}$ will stand for the corresponding operator norm. Below, in connection with the finite element space, we shall also need to use a special operator norm which will be specified additionally. In what follows we shall sometimes work with functions defined on domains different from Ω (or $\overline{\Omega}$). We therefore denote by $\|.\|_{p;\mathcal{D}}$, $\|.\|_{p;l;\mathcal{D}}$, and $|.|_{\xi;\mathcal{D}}$, the norms in the spaces $L_p(\mathcal{D})$, $W^l_p(\mathcal{D})$, and $\mathcal{C}^\xi(\overline{\mathcal{D}})$, respectively (identifying, as above, equivalent norms). For our subsequent needs we also denote $(v, w)_\mathcal{D} = \int_\mathcal{D} v\overline{w}\, dx$, and we shall write $(.,.)$ instead of $(.,.)_\Omega$.

As already mentioned above, for elliptic partial differential operators there are known resolvent estimates in different function spaces which allow one to show that the above operator A generates a holomorphic semigroup e^{-tA} in these spaces. Such estimates are usually stated under quite general restrictions on the operator in question and, as a rule, the sector of analyticity of the semigroup e^{-tA} is not specified. For our more concrete situation it is possible to derive refined estimates, which are given below. Moreover, we generalize these results and present as well related estimates in pairs of spaces.

For finite element versions of elliptic operators resolvent estimates can be easily derived in L_2 norm but it is not a simple problem to show them in the case of L_p norm, $p \neq 2$. Note that if one has a suitable estimate in L_∞ norm, by duality and interpolation it immediately extends to the whole scale of Lebesgue norms $\|.\|_p$, $1 \leq p \leq \infty$. It is worth mentioning that the problem in the case of L_∞ norm is brought to an end at least for second order operators with real-valued sufficiently smooth coefficients due to [4] (the case with at least quadratic elements) and [7] (the case with linear elements) in the sense that these results yield a uniform resolvent estimate in L_∞ norm which is valid in any closed sector outside of the real positive semi-axis. We are not however informed of any estimates in Hölder norms or estimates involving two different norms. In particular, for the time being, no uniform estimate of the gradient is known in L_∞ norm. In what follows we show such estimates in the case of finite element discretization with piecewise linear test functions. As concerns the case with higher degree elements, it seems to be simpler for analysis for the reason that the L_∞ norm of the Ritz projection is uniformly bounded in this situation (unlike in the case with linear elements; cf. [30]).

Throughout this work we shall denote by C and c generic constants, subject to $C \geq 0$ and $c > 0$, whose sizes will be unessential for our subsequent analysis.

Let now $h \in (0, h_0]$, $h_0 > 0$, be a small parameter and let $\mathcal{T}_h = \{\tau_j\}_{j=1}^{J_h}$,

be a triangulation of $\Omega_h = \text{Int}\,(\cup_j \overline{\tau_j}) \subset \Omega$ into mutually disjoint open face-to-face simplices τ_j. It will be convenient to take $h = \max_j \text{diam}\,\tau_j$. We assume that the vertices of simplices τ_j which belong to $\partial\Omega_h$ lie also on $\partial\Omega$ and that the family $\{\mathcal{T}_h\}$ of triangulations is globally quasi-uniform in the sense that $\min_j \text{vol}\,(\tau_j) \geq ch^d$. It follows from our assumptions that $\text{dist}\,(x, \partial\Omega) \leq Ch^2$ for $x \in \partial\Omega_h$. Let further S_h be the finite dimensional space of all continuous complex-valued piecewise linear functions, associated with $\mathcal{T}_h$, that vanish outside Ω_h. To the operator A we associate its finite element version $A_h : S_h \to S_h$ by

$$(A_h\psi, \chi) = (\nabla\psi, \nabla\chi) \quad \text{for all } \psi, \chi \in S_h.$$

If we consider the parabolic initial-boundary value problem

$$u_t + Au = 0, \quad t > 0; \quad u(0) = v, \tag{1.1}$$

associated to the operator A, its solution is given, with the aid of the semigroup e^{-tA}, by $u(t) = e^{-tA}v$. Then a spatial semi-discrete approximation to $u(t)$ may be taken as $u_h(t) = e^{-tA_h}v_h$, with v_h a suitable approximation to v in (1.1). For any $h \in (0, h_0]$ the operator A_h is bounded which yields that the exponential function e^{-tA_h} is a bounded operator as well (for any fixed $h \in (0, h_0]$ and $t > 0$). The known resolvent estimates allow one to show that e^{-tA_h} is in fact uniformly bounded with respect to $h \in (0, h_0]$ and $t \geq 0$, in the operator norms of the corresponding function spaces. This fact is most essential for the analysis of stability and convergence of spatial semi-discrete approximations to (1.1). Moreover, for one step methods approximating (1.1) both in space and in time

$$U^n = r(-kA_h)^n v_h, \tag{1.2}$$

where $r(z)$ is a rational function (it just specifies the method), the stability of U^n and its convergence to the solution of (1.1) can be shown by making use of the same resolvent estimates for the operator A_h (cf., e.g., Bakaev [3] or Thomée [32]). It is worth mentioning that, for the analysis of fully discrete approximations based on the application of A(φ)-stable methods, it is essential to use resolvent estimates of the operator A_h that would be valid outside of any closed sector around the real positive semi-axis; just such estimates were found in L_∞ norm in [4] and [7]. Using duality and interpolation arguments it is possible to extend the results of [4] and [7] to the whole scale L_p, $1 \leq p \leq \infty$ of Lebesgue norms. In this work we show resolvent estimates of A_h in (Lebesgue, Lebesgue), (Hölder, Lebesgue), and (Hölder, Hölder) pairs of norms; all of them are valid outside of any sector containing the real positive semi-axis. In particular, a uniform resolvent estimate in L_∞ norm for the gradient can be obtained as a direct consequence of Theorem 3.5 below. Our results can be used, for example, for showing the stability and convergence of U^n in (1.2) in the corresponding pairs of of norms. For further possible applications of the below estimates we can refer to [5].

Our main results are collected in the following two sections. In Section 2 we show resolvent estimates in pairs of spaces for the operator A. These assertions are further applied in Section 3 in order to obtain analogous estimates for the finite element operator A_h. All the restrictions stated above are assumed to be in force throughout this work. The below technical tools are similar to those used in [4] and [7]; they are based on comparing the discrete resolvent to the continuous one and on making use of estimates for the continuous problem.

2 RESOLVENT ESTIMATES OF THE OPERATOR A

In this section we present, as mentioned above, resolvent estimates for the elliptic differential operator A in pairs of function spaces. It will be convenient, given $\varphi \in (0, \frac{\pi}{2})$, to denote

$$\Sigma_\varphi = \{\lambda : \lambda \neq 0, |\arg \lambda| \leq \varphi\} \cup \{0\}.$$

We start with showing some auxiliary results of which the first will be related to real interpolation theory.

Let $\left(\dot{\mathcal{C}}^0, \dot{W}^1_\infty\right)_{\theta,\infty}$, $0 < \theta < 1$, be the interpolation space between $\dot{\mathcal{C}}^0$ and $\dot{W}^1_\infty$ constructed by the K-method (cf., e.g., Triebel [35, pp.23-24]). The norm in $\left(\dot{\mathcal{C}}^0, \dot{W}^1_\infty\right)_{\theta,\infty}$ is given by

$$\|v\|_{\left(\dot{\mathcal{C}}^0, \dot{W}^1_\infty\right)_{\theta,\infty}} = \sup_{s>0} s^{-\theta} \inf_{\substack{v=w_0(s)+w_1(s),\\ w_0(s)\in\dot{\mathcal{C}}^0, w_1(s)\in\dot{W}^1_\infty}} (|w_0(s)|_0 + s\|w_1(s)\|_{\infty;1}).$$

Lemma 2.1. *For any fixed $0 < \alpha < 1$, we have*

$$\|v\|_{\left(\dot{\mathcal{C}}^0, \dot{W}^1_\infty\right)_{\alpha,\infty}} \leq C|v|_\alpha \quad \textit{for all } v \in \dot{\mathcal{C}}^\alpha. \tag{2.1}$$

This result can be shown by direct use of the definition of $\|v\|_{\left(\dot{\mathcal{C}}^0, \dot{W}^1_\infty\right)_{\alpha,\infty}}$ and by application of standard methods in real interpolation theory. We also note that the same estimate with $\dot{\mathcal{C}}^1$ in place of $\dot{W}^1_\infty$ follows from Lunardi [21, Proposition 2.6 and Remark 2.9], but the norm in $\left(\dot{\mathcal{C}}^0, \dot{W}^1_\infty\right)_{\alpha,\infty}$ is dominated by that in $\left(\dot{\mathcal{C}}^0, \dot{\mathcal{C}}^1\right)_{\alpha,\infty}$.

Let further $G(s; x, y)$, $s > 0$, be the Green's function for the operator $(sI + A)^{-1}$. We shall now obtain some pointwise estimates which will be quite useful in the sequel.

Lemma 2.2. *For any fixed $0 \leq \alpha < \frac{2}{d}$ ($0 \leq \alpha \leq 1$ if $d = 1$), we have*

$$\begin{aligned} |G(s; x, y)| \;\leq\; & C\frac{s^{\frac{d\alpha}{2}-1}}{2d^{-1} - \alpha} \cdot \frac{\exp(-\sqrt{s}|x-y|)}{|x-y|^{d(1-\alpha)}} + C(s+1)^{-1}, \\ & \textit{for all } x, y \in \Omega \textit{ and } s > 0. \end{aligned} \tag{2.2}$$

Furthermore, if α is restricted by $0 \le \alpha < \frac{2}{d+1}$, then for all $x, y \in \Omega$ and $s > 0$,

$$|\nabla_x G(s;x,y)| \le C \frac{s^{\frac{(d+1)\alpha}{2}-1}}{2(d+1)^{-1}-\alpha} \cdot \frac{\exp(-\sqrt{s}|x-y|)}{|x-y|^{(d+1)(1-\alpha)}} + C(s+1)^{-1}. \quad (2.3)$$

Proof. Let us denote for short $\rho = |x-y|$. For $\mathfrak{G}(t;x,y)$ the Green's function of the related parabolic problem (1.1), we have (cf. [17])

$$|\mathfrak{G}(t;x,y)| \le C(t+\rho^2)^{-\frac{d}{2}} \exp(-c\rho^2/t) + C\exp(-ct), \quad t > 0. \quad (2.4)$$

Now, in order to show (2.2), we use the representation

$$G(s;x,y) = \int_0^\infty e^{-st} \mathfrak{G}(t;x,y)\, dt,$$

which yields with the aid of (2.4), since $t + \rho^2 \ge t^\alpha \rho^{2(1-\alpha)}$,

$$\begin{aligned} |G(s;x,y)| &\le C \int_0^\infty (t+\rho^2)^{-\frac{d}{2}} \exp(-st - c\rho^2/t)\, dt + \frac{C}{s+1} \\ &\le C s^{\frac{d\alpha}{2}-1} \rho^{-(1-\alpha)d} \int_0^\infty t^{-\frac{d\alpha}{2}} \exp(-t - cs\rho^2/t)\, dt + \frac{C}{s+1}. \end{aligned}$$

The integral on the right-hand side of the last estimate converges for $0 \le \alpha < \frac{2}{d}$, and we see that (2.2) follows by using the evident estimate

$$\exp(-t/2 - cs\rho^2/t) \le \exp(-c\sqrt{s}\rho).$$

The second stated inequality (2.3) can be shown similarly by applying the estimate (see, e.g., [17])

$$|\nabla_x \mathfrak{G}(t;x,y)| \le C(t+\rho^2)^{-\frac{d+1}{2}} \exp(-c\rho^2/t) + C\exp(-ct), \quad t > 0. \quad (2.5)$$

(Note that (2.3) formally obtains by comparing (2.5) to (2.4) and substituting $(d+1)$ for d into (2.2)). □

Now we turn to showing resolvent estimates themselves. The first result is not original but it is needed for our subsequent purposes.

Theorem 2.1. *For any fixed $\varphi \in (0, \frac{\pi}{2})$, we have*

$$\begin{aligned} &|(\lambda I - A)^{-1} v|_j \le C(1+|\lambda|)^{-1+\frac{j}{2}} |v|_0, \\ &\text{for } \lambda \notin \mathit{Int}\, \Sigma_\varphi,\ j = 0, 1, \text{ and } v \in L_\infty, \end{aligned} \quad (2.6)$$

and

$$|A(\lambda I - A)^{-1} v|_0 \le C(1+|\lambda|)^{-\frac{1}{2}} |v|_1 \quad \text{for } \lambda \notin \mathit{Int}\, \Sigma_\varphi \text{ and } v \in \dot{W}^1_\infty. \quad (2.7)$$

In principle (2.6) is an almost direct consequence of a quite general result of Stewart [31] which has been sharpened in [7]. In this work we accept in fact just the same starting assumptions as in [7] (which are more restrictive than in [31]) and we take the resolvent estimate (2.6) itself actually in the same form as it is stated in [7]. The minor difference is however that (2.6) is stated in [7] only for $v \in \mathcal{C}$, but, since the resolvent is an integral operator, the result is clearly still valid for $v \in L_\infty$. For a proof of (2.7), see [7].

In what follows we shall also give analogues of (2.6) and (2.7) in L_p norm. Moreover, an analogue of (2.6) with $j = 0$ can be stated right now:

Theorem 2.2. *For any fixed $\varphi \in (0, \frac{\pi}{2})$, we have*

$$\begin{aligned} &\|(\lambda I - A)^{-1} v\|_p \le C(1 + |\lambda|)^{-1} \|v\|_p, \\ &\text{for } 1 \le p \le \infty,\ \lambda \notin \mathit{Int}\ \Sigma_\varphi,\ \text{and } v \in L_p. \end{aligned} \tag{2.8}$$

In fact (2.8) with $p = 1$ follows from (2.6) with $j = 0$ by duality and further the result extends, by interpolation, to the general case $1 \le p \le \infty$.

We shall also need to use below some estimates in Hölder norms.

Theorem 2.3. *For any fixed $\varphi \in (0, \frac{\pi}{2})$ and for $\alpha = 0, 1$,*

$$\begin{aligned} &|(\lambda I - A)^{-1} v|_{1+\alpha} \le C(1 + |\lambda|)^{-1+\frac{\alpha}{2}} |v|_1, \\ &\text{for all } \lambda \notin \mathit{Int}\ \Sigma_\varphi \text{ and } v \in \dot{W}^1_\infty. \end{aligned} \tag{2.9}$$

Proof. We shall first show that the assertion is true if $|\lambda| \ge R$, with $R > 0$ sufficiently large. We start with taking $\alpha = 0$. Given $\lambda \notin \text{Int } \Sigma_\varphi$, we let $\mu = -|\lambda|$. Applying now Theorem 3.2 in [12] with $p \to \infty$ and fixing $R > 0$ sufficiently large, we find that (2.9) with $\alpha = 0$ holds with μ written for λ, for $|\mu| \ge R$. We therefore have, since $|\mu| = |\lambda|$,

$$|(\mu I - A)^{-1} v|_1 \le C(1 + |\mu|)^{-1} |v|_1 = C(1 + |\lambda|)^{-1} |v|_1 \quad \text{for } |\lambda| \ge R. \tag{2.10}$$

At the same time, it follows from (2.6) with $j = 1$ and (2.7) that, for $|\lambda| \ge R$,

$$\begin{aligned} |(\mu I - A)^{-1} A\,(\lambda I - A)^{-1} v|_1 &\le C(1 + |\mu|)^{-\frac{1}{2}} |A(\lambda I - A)^{-1} v|_0 \\ &\le C(1 + |\lambda|)^{-1} |v|_1. \end{aligned} \tag{2.11}$$

Next in view of the identity

$$\lambda(\lambda I - A)^{-1} = I + A(\lambda I - A)^{-1}, \tag{2.12}$$

a simple calculation shows, since $|\mu| = |\lambda|$,

$$\begin{aligned} |(\mu I - A)^{-1} \mu\,(\lambda I - A)^{-1} v|_1 &= |(\mu I - A)^{-1} \lambda\,(\lambda I - A)^{-1} v|_1 \\ &\le |(\mu I - A)^{-1} v|_1 + |(\mu I - A)^{-1} A\,(\lambda I - A)^{-1} v|_1. \end{aligned} \tag{2.13}$$

Using (2.10), (2.11), (2.13), the identity

$$(\lambda I - A)^{-1} = (\mu I - A)^{-1} (\mu I - A)\,(\lambda I - A)^{-1}, \tag{2.14}$$

and the fact that $|\mu| = |\lambda|$, we thus obtain for $|\lambda| \geq R$,

$$\begin{aligned} |(\lambda I - A)^{-1} v|_1 &\leq |(\mu I - A)^{-1} \mu \, (\lambda I - A)^{-1} v|_1 \\ &\quad + |(\mu I - A)^{-1} A \, (\lambda I - A)^{-1} v|_1 \\ &\leq C(1 + |\lambda|)^{-1} |v|_1. \end{aligned} \tag{2.15}$$

This yields that (2.9) with $\alpha = 0$ holds for $|\lambda| \geq R$.

Applying further Theorem 3.2 in [12] with $s = 1$ and using again (2.14), we get

$$\begin{aligned} |(\lambda I - A)^{-1} v|_2 &\leq |(\mu I - A)^{-1} (\mu - \lambda) \, (\lambda I - A)^{-1} v|_2 \\ &\quad + |(\mu I - A)^{-1} v|_2 \\ &\leq C(1 + |\mu|)^{-\frac{1}{2}} \left(|(\mu - \lambda) \, (\lambda I - A)^{-1} v|_1 + |v|_1 \right), \end{aligned} \tag{2.16}$$

and with the aid of the estimate (2.9) with $\alpha = 0$ (already proved) we see that in fact (2.9) holds also with $\alpha = 1$ if $\lambda \geq R$ and $R > 0$ is sufficiently large.

It remains to show that (2.9) is also in force for $\lambda \leq R$, with any fixed $R \geq 0$. Note that it would suffice in fact to prove that for $\alpha = 0, 1$,

$$|(\lambda I - A)^{-1} v|_{1+\alpha} \leq C |v|_1 \quad \text{for } \lambda \notin \text{Int } \Sigma_\varphi, \ |\lambda| \leq R, \text{ and } v \in \dot{W}^1_\infty. \tag{2.17}$$

For $\alpha = 0$, the last estimate immediately follows by (2.6) with $j = 1$, since $|v|_0 \leq C|v|_1$. In order to show that (2.17) holds as well for $\alpha = 1$, we write, using well-known inequalities between Hölder norms, the estimate $|w|_{5/2} \leq C|Aw|_{1/2}$ (cf. [2]), and the above identity (2.12),

$$\begin{aligned} |(\lambda I - A)^{-1} v|_2 &\leq C |(\lambda I - A)^{-1} v|_{5/2} \\ &\leq C |A(\lambda I - A)^{-1} v|_{1/2} \\ &\leq C \left(|v|_1 + |\lambda| \, |(\lambda I - A)^{-1} v|_1 \right). \end{aligned}$$

Now combining this and (2.17) with $\alpha = 0$ (already shown) leads easily to (2.17) with $\alpha = 1$.

This completes the proof. □

We are ready now to show resolvent estimates in pairs of Lebesgue spaces. We shall state first a result for the particular case when $\lambda = -s$, $s > 0$, and when one of the spaces is L_∞.

Theorem 2.4. *For any fixed $\frac{d}{2} < q \leq \infty$ (for $1 \leq q \leq \infty$ if $d = 1$), we have*

$$\|(sI + A)^{-1} v\|_\infty \leq C(s+1)^{\frac{d}{2q} - 1} \|v\|_q \quad \textit{for all } s \geq 0 \textit{ and } v \in L_q. \tag{2.18}$$

Proof. A suitable argument somewhat changes in details for $d = 1$ and for $d \geq 2$ and one has to distinguish between these cases. The case $d = 1$ is however simpler and we shall further concentrate on the situation when $d \geq 2$.

Assuming therefore that $d \geq 2$, we select a fixed α such that $q > \frac{1}{\alpha} > \frac{d}{2}$. Defining further q' by $\frac{1}{q'} := 1 - \frac{1}{q}$ and noting that $0 < \alpha < \frac{2}{d}$ and $d - dq'(1-\alpha) > 0$, we apply (2.2) to obtain for all $v \in L_q$ with $\|v\|_q = 1$ (as above we denote $\rho = |x-y|$), if $s > 0$,

$$\begin{aligned}
\|(sI+A)^{-1}v\|_\infty &\leq \sup_{x\in\Omega} \left(\int_\Omega |G(s;x,y)|^{q'}\, dy \right)^{1/q'} \\
&\leq C s^{\frac{d\alpha}{2}-1} \sup_{x\in\Omega} \left(\int_\Omega \frac{\exp(-cq'\sqrt{s}\rho)}{\rho^{dq'(1-\alpha)}}\, dy \right)^{1/q'} \\
&\leq C s^{\frac{d\alpha}{2}-1} \left(\int_0^\infty \exp(-cq'\sqrt{s}z)\, z^{d-1-dq'(1-\alpha)}\, dz \right)^{1/q'} \\
&\leq C s^{\frac{d}{2q}-1},
\end{aligned}$$

which shows the claim at least for $s \geq 1$.

It thus remains to prove (2.18) for $0 \leq s \leq 1$. Clearly, it would suffice to show instead that

$$\|(sI+A)^{-1}v\|_\infty \leq C\|v\|_q \quad \text{for } 0 \leq s \leq 1.$$

The last estimate follows however easily from (2.12), (2.8) with $p = \infty$ and $\lambda = -s$, and the inequality $\|A^{-1}v\|_\infty \leq C\|v\|_q$ (valid for $q > \frac{d}{2}$) with the aid of

$$\|(sI+A)^{-1}v\|_\infty = \|A(sI+A)^{-1}A^{-1}v\|_\infty \leq C\|A^{-1}v\|_\infty.$$

This completes the proof. □

Now we can obtain sectorial estimates in (L_p, L_q) pairs.

Theorem 2.5. *For any fixed $\varphi \in (0, \frac{\pi}{2})$ and $1 \leq q \leq p \leq \infty$ such that $\frac{1}{q} - \frac{1}{p} < \frac{2}{d}$ (for any fixed $1 \leq q \leq p \leq \infty$ if $d = 1$), we have*

$$\begin{aligned}
&\|(\lambda I - A)^{-1}v\|_p \leq C(1+|\lambda|)^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-1}\|v\|_q, \\
&\qquad \text{for all } \lambda \notin \mathit{Int}\, \Sigma_\varphi \text{ and } v \in L_q.
\end{aligned} \tag{2.19}$$

Proof. As above, we shall restrict our consideration to the case $d \geq 2$.

Assuming first that $\frac{1}{q} < \frac{2}{d}$, by (2.12), (2.8) with $p = \infty$, and (2.18), we find for all $\lambda \notin \text{Int } \Sigma_\varphi$,

$$\begin{aligned}
\|(\lambda I - A)^{-1}v\|_\infty &= \|(|\lambda| I + A)\,(\lambda I - A)^{-1}(|\lambda| I + A)^{-1}v\|_\infty \\
&\leq \left(1 + 2|\lambda|\, \|(\lambda I - A)^{-1}\|_{L_\infty \to L_\infty}\right) \|(|\lambda| I + A)^{-1}v\|_\infty \\
&\leq C\|(|\lambda| I + A)^{-1}v\|_\infty \leq C(1+|\lambda|)^{\frac{d}{2q}-1}\|v\|_q,
\end{aligned}$$

which shows the claim for $p = \infty$ and $\frac{1}{q} < \frac{2}{d}$. Observe also that, by duality, this yields in turn that (2.19) holds as well for $q = 1$ and $p \geq 1$ such that $1 - \frac{1}{p} < \frac{2}{d}$.

Let us now consider the general case with arbitrary p and q such that $\frac{1}{q} - \frac{1}{p} < \frac{2}{d}$. Define q_0 and p_1 by

$$\frac{1}{q_0} := \frac{1}{q} - \frac{1}{p} =: 1 - \frac{1}{p_1}.$$

Noting that $\frac{1}{q_0} = 1 - \frac{1}{p_1} < \frac{2}{d}$, we see that in view of the above reasonings (2.19) holds for $p = \infty$, $q = q_0$ and for $p = p_1$, $q = 1$. For arbitrary p and q, with $\frac{1}{q} - \frac{1}{p} < \frac{2}{d}$, the desired result thus obtains by interpolation. □

Further we shall obtain resolvent estimates in (Hölder, Lebesgue) pairs of spaces. As above, we shall consider first the particular case when $\lambda = -s$, $s \geq 0$.

Theorem 2.6. *For any fixed $0 \leq \xi \leq 1$ and $q \in \left(\frac{d}{2-\xi}, \infty\right]$, we have*

$$|(sI + A)^{-1} v|_\xi \leq C(s+1)^{\frac{\xi}{2} + \frac{d}{2q} - 1} \|v\|_q \quad \textit{for all } s \geq 0 \textit{ and } v \in L_q. \tag{2.20}$$

Proof. Note that, in the case $\xi = 1$ and $q > d$, the result obtains by using (2.3) and the estimate $|A^{-1}v|_1 \leq C\|v\|_q$ (cf. the proof of Theorem 2.4). In particular, (2.20) holds for $\xi = 1$ and $q = \infty$. Since, by (2.8) with $p = \infty$, it holds as well for $\xi = 0$ and $q = \infty$, by interpolation, we have shown the result in fact for $0 \leq \xi \leq 1$ and $q = \infty$.

Let us turn to the general case. Let $\xi \in [0, 1]$ and $q \in \left(\frac{d}{2-\xi}, \infty\right]$ be fixed. We can write, with some $\varepsilon > 0$,

$$\frac{1}{q} = \frac{2 - \xi}{d} - \varepsilon. \tag{2.21}$$

Assuming that $\varepsilon < \frac{1}{d}$ and defining then q_0 and q_1 by

$$\frac{1}{q_0} := \frac{2}{d} - \varepsilon, \quad \frac{1}{q_1} := \frac{1}{d} - \varepsilon,$$

we conclude that (2.20) holds for $\xi = 0$ and $q = q_0$ by (2.18) with $q = q_0$, and it will hold, as already shown, for $\xi = 1$ and $q = q_1$, since $q_1 > d$. The desired result in the general case will thus follow by interpolation.

If it happens that, in (2.21), $\varepsilon \geq \frac{1}{d}$, which means that q is sufficiently large, the result nevertheless will follow by interpolation, since we have already shown that it is in force for $q = \infty$ and for q sufficiently close to $\frac{d}{2-\xi}$. □

Sectorial estimates in $(\mathcal{C}^\xi, L_q)$ pairs will then obtain as follows:

Theorem 2.7. *For any fixed $\varphi \in (0, \frac{\pi}{2})$, $0 \leq \xi \leq 1$, and $q \in \left(\frac{d}{2-\xi}, \infty\right]$, we have*

$$|(\lambda I - A)^{-1} v|_\xi \leq C(1 + |\lambda|)^{\frac{\xi}{2} + \frac{d}{2q} - 1} \|v\|_q,$$
$$\textit{for all } \lambda \notin \textit{Int } \Sigma_\varphi \textit{ and } v \in L_q. \tag{2.22}$$

Proof. It follows from (2.12) and (2.8) with $p = q$ that for all $\lambda \notin \text{Int } \Sigma_\varphi$,

$$\|(|\lambda| I + A)\,(\lambda I - A)^{-1} v\|_q \le \|v\|_q + 2|\lambda|\,\|(\lambda I - A)^{-1} v\|_q \le C\|v\|_q.$$

Using this thus yields

$$\begin{aligned} |(\lambda I - A)^{-1} v|_\xi &\le \|\!|(|\lambda| I + A)^{-1}|\!\|_{L_q \to \mathcal{C}^\xi} \|(|\lambda| I + A)\,(\lambda I - A)^{-1} v\|_q \\ &\le C\|\!|(|\lambda| I + A)^{-1}|\!\|_{L_q \to \mathcal{C}^\xi} \|v\|_q. \end{aligned}$$

It remains to apply (2.20) with $s = |\lambda|$. □

We shall next show analogues of (2.6) with $j = 1$ and of (2.7) in L_p norm.

Theorem 2.8. *For any fixed $\varphi \in (0, \frac{\pi}{2})$ and $1 \le q \le p \le \infty$ such that $\frac{1}{q} - \frac{1}{p} < \frac{1}{d}$, we have for all $\lambda \notin$ Int Σ_φ and $v \in L_q$,*

$$\|(\lambda I - A)^{-1} v\|_{p;1} \le C(1 + |\lambda|)^{\frac{d}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{1}{2}} \|v\|_q. \tag{2.23}$$

Proof. We can write, with some $\varepsilon \in (0, \frac{1}{d}]$,

$$\frac{1}{q} - \frac{1}{p} = \frac{1}{d} - \varepsilon.$$

Let further q_1 and p_0 be defined by

$$\frac{1}{q_1} := \frac{1}{d} - \varepsilon, \quad 1 - \frac{1}{p_0} := \frac{1}{d} - \varepsilon.$$

By (2.20) with $\xi = 1$, we have, since $\frac{1}{q_1} < \frac{1}{d}$,

$$\|(sI + A)^{-1} v\|_{\infty;1} \le C(s+1)^{\frac{d}{2q_1} - \frac{1}{2}} \|v\|_{q_1} \quad \text{for all } s \ge 0 \text{ and } v \in L_{q_1}. \tag{2.24}$$

We define now q_0 by $\frac{1}{q_0} := 1 - \frac{1}{p_0}$ (note that $\frac{1}{q_0} < \frac{1}{d}$) and select some $\alpha \in \left(\frac{1}{d+1}(1 + \frac{d}{q_0}), \frac{2}{d+1}\right)$. Then

$$(\nabla (sI + A)^{-1} v, \psi) = \int_\Omega \int_\Omega \overline{\psi(x)}\, \nabla_x G(s; x, y)\, v(y)\, dy\, dx,$$

which yields, using (2.3) and Hölder's inequality, for all $s > 0$, $v \in L_1$, and $\psi \in L_{q_0}$,

$$\begin{aligned} |(\nabla (sI + A)^{-1} v, \psi)| &\le \|v\|_1 \sup_{y \in \Omega} \int_\Omega |\nabla_x G(s; x, y)|\, |\psi(x)|\, dx \\ &\le C s^{\frac{d}{2q_0} - \frac{1}{2}} \|v\|_1 \|\psi\|_{q_0}. \end{aligned}$$

This implies in turn

$$\begin{aligned} \|(sI + A)^{-1} v\|_{p_0;1} \le C(s+1)^{\frac{d}{2}(1 - \frac{1}{p_0}) - \frac{1}{2}} \|v\|_1, \\ \text{for all } s \ge 1 \text{ and } v \in L_1. \end{aligned} \tag{2.25}$$

As above (cf. the argument used in the proof of Theorem 2.4), the restriction $s \geq 1$ can be replaced by $s \geq 0$.

By interpolation, we obtain from (2.24) and (2.25),

$$\|(sI + A)^{-1}v\|_{p;1} \leq C(s+1)^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}\|v\|_q \quad \text{for all } s \geq 0 \text{ and } v \in L_q.$$

Using this finally yields for all $\lambda \notin \text{Int } \Sigma_\varphi$ and $v \in L_q$, with the aid of (2.12),

$$\begin{aligned}\|(\lambda I - A)^{-1}v\|_{p;1} &= \|(|\lambda|I + A)^{-1}(|\lambda|I + A)\,(\lambda I - A)^{-1}v\|_{p;1} \\ &\leq C(1+|\lambda|)^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}\|(|\lambda|I + A)\,(\lambda I - A)^{-1}v\|_q \\ &\leq C(1+|\lambda|)^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}\left(|\lambda|\,\|(\lambda I - A)^{-1}v\|_q + \|v\|_q\right),\end{aligned}$$

whence the claim follows by (2.8) with q substituted for p. □

Theorem 2.9. *For any fixed $\varphi \in (0, \frac{\pi}{2})$, we have*

$$\begin{aligned}&\|A(\lambda I - A)^{-1}v\|_p \leq C(1+|\lambda|)^{-\frac{1}{2}}\|v\|_{p;1}, \\ &\textit{for all } 1 \leq p \leq \infty,\ \lambda \notin \textit{Int } \Sigma_\varphi,\ \textit{and } v \in W_p^1.\end{aligned} \tag{2.26}$$

Proof. For all $\psi \in L_{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$, we can write

$$(A(\lambda I - A)^{-1}v, \psi) = (\nabla v, \nabla(\overline{\lambda} I - A)^{-1}\psi),$$

whence, in view of (2.23) with $p = q$, in which we substitute p' for p,

$$|(A(\lambda I - A)^{-1}v, \psi)| \leq C\|v\|_{p;1}(1+|\lambda|)^{-1/2}\|\psi\|_{p'},$$

which shows the claim. □

Now we shall give some generalizations of our above results which will be expressed in terms of estimates involving the iterated resolvent operator.

Theorem 2.10. *Let $\xi \in [0,1]$, $q \in [1,\infty]$, and $m \in \mathbb{N} \cup \{0\}$ be such that*

$$\frac{1}{q} < \frac{2m + 2 - \xi}{d}. \tag{2.27}$$

Then for any fixed $\varphi \in (0, \frac{\pi}{2})$,

$$\begin{aligned}&|(\lambda I - A)^{-(m+1)}v|_\xi \leq C(1+|\lambda|)^{\frac{\xi}{2}+\frac{d}{2q}-(m+1)}\|v\|_q, \\ &\textit{for all } \lambda \notin \textit{Int } \Sigma_\varphi \textit{ and } v \in L_q.\end{aligned} \tag{2.28}$$

Proof. Let $q_0, q_1, \ldots, q_{m-1}$ be chosen such that

$$q \leq q_{m-1} \leq \ldots \leq q_0 \leq \infty,$$

and

$$\begin{aligned} \frac{1}{q_0} &< \frac{2-\xi}{d}, \\ \frac{1}{q_1}-\frac{1}{q_0} &< \frac{2}{d}, \\ &\cdots \quad \cdots \quad \cdots \\ \frac{1}{q}-\frac{1}{q_{m-1}} &< \frac{2}{d}. \end{aligned}$$

Then applying (2.22) with q_0 substituted for q and using further repeatedly (2.19), we get

$$\begin{aligned} |(\lambda I-A)^{-(m+1)}v|_\xi &\le C(1+|\lambda|)^{\frac{\xi}{2}+\frac{d}{2q_0}-1}\|(\lambda I-A)^{-m}v\|_{q_0} \\ &\le C(1+|\lambda|)^{\frac{\xi}{2}+\frac{d}{2q_1}-2}\|(\lambda I-A)^{-(m-1)}v\|_{q_1} \\ &\le \ldots \le C(1+|\lambda|)^{\frac{\xi}{2}+\frac{d}{2q}-(m+1)}\|v\|_q, \\ &\quad \text{for all } \lambda \notin \text{Int } \Sigma_\varphi \text{ and } v \in L_q, \end{aligned}$$

which is the desired result. □

Theorem 2.11. *Let $m \in \mathbb{N} \cup \{0\}$ and $1 \le q \le p \le \infty$ be such that*

$$\frac{1}{q}-\frac{1}{p} < \frac{2m}{d}.$$

Then for any fixed $\varphi \in (0, \frac{\pi}{2})$,

$$\begin{aligned} \|(\lambda I-A)^{-m}v\|_p \le C(1+|\lambda|)^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-m}\|v\|_q, \\ \text{for all } \lambda \notin \text{Int } \Sigma_\varphi \text{ and } v \in L_q. \end{aligned} \tag{2.29}$$

Theorem 2.12. *Let $m \in \mathbb{N} \cup \{0\}$ and $1 \le q \le p \le \infty$ be such that*

$$\frac{1}{q}-\frac{1}{p} < \frac{2m+1}{d}.$$

Then for any fixed $\varphi \in (0, \frac{\pi}{2})$,

$$\begin{aligned} \|(\lambda I-A)^{-(m+1)}v\|_{p;1} \le C(1+|\lambda|)^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-(m+\frac{1}{2})}\|v\|_q, \\ \text{for all } \lambda \notin \text{Int } \Sigma_\varphi \text{ and } v \in L_q. \end{aligned} \tag{2.30}$$

The proofs of Theorems 2.11 and 2.12 are similar to that of Theorem 2.10. They are based on using (2.19) and (2.23).

It remains to show resolvent estimates in pairs of Hölder spaces.

Theorem 2.13. *For any fixed $\varphi \in (0, \frac{\pi}{2})$, $\xi \in [0,1]$, and $\eta \in [0, \xi]$, we have*

$$|(\lambda I-A)^{-1}v|_\xi \le C(1+|\lambda|)^{\frac{\xi-\eta}{2}-1}|v|_\eta \quad \text{for all } \lambda \notin \text{Int } \Sigma_\varphi \text{ and } v \in \dot{\mathcal{C}}^\eta. \tag{2.31}$$

In the case $\eta = 0$, the result is still valid for all $v \in L_\infty$, and in the case $\xi = \eta = 1$, it is valid for all $v \in \dot{W}^1_\infty$.

Proof. In the case $\eta = 0$, (2.31) holds for all $v \in L_\infty$ at least if $\xi = 0$ or $\xi = 1$. This follows directly from (2.6). By interpolation, we get immediately for $\xi \in [0,1]$,

$$|(\lambda I - A)^{-1} v|_\xi \le C(1+|\lambda|)^{\frac{\xi}{2}-1} |v|_0 \quad \text{for all } v \in L_\infty, \tag{2.32}$$

which shows the claim in the case $\eta = 0$.

Next note that the result also holds in the case $\xi = \eta = 1$, for all $v \in \dot{W}^1_\infty$, as follows from (2.9) with $\alpha = 0$. In particular, (2.31) with $\xi = \eta = 1$ is true for all $v \in \dot{\mathcal{C}}^1$.

Moreover, with this in mind and using (2.32) with $\xi = 0$ (which clearly can be considered for all $v \in \dot{\mathcal{C}}$), we get by interpolation (see, e.g., Triebel [35, Theorem 1.3.3]), for $\xi \in (0,1)$,

$$|(\lambda I - A)^{-1} v|_\xi \le C(1+|\lambda|)^{-1} \|v\|_{\left(\dot{\mathcal{C}}, \dot{W}^1_\infty\right)_\xi} \quad \text{for all } v \in \dot{\mathcal{C}}^\xi,$$

whence, applying (2.1) yields, if $\xi \in (0,1)$,

$$|(\lambda I - A)^{-1} v|_\xi \le C(1+|\lambda|)^{-1} |v|_\xi \quad \text{for all } v \in \dot{\mathcal{C}}^\xi. \tag{2.33}$$

The last result can be also thought of as a refined version of a well-known resolvent estimate in Hölder norms (cf. Campanato [8] and Cannarsa, Terreni & Vespri [9]). Note that, in view of the above comment, (2.33) also holds for $\xi = 0$ and $\xi = 1$.

On the other hand, applying (2.9) with $\alpha = 0$ and (2.32) with $\xi = 1$ (which clearly can be considered for all $v \in \dot{\mathcal{C}}$) and using again an interpolation argument combined with (2.1), we get, for $0 \le \eta \le 1$,

$$|(\lambda I - A)^{-1} v|_1 \le C(1+|\lambda|)^{\frac{1-\eta}{2}-1} |v|_\eta \quad \text{for all } v \in \dot{\mathcal{C}}^\eta. \tag{2.34}$$

(This is a direct consequence of (2.32) with $\xi = 1$ in the case $\eta = 0$ and of (2.9) with $\alpha = 0$ in the case $\eta = 1$).

Finally, we get by interpolation from (2.33) and (2.34), for $\xi \in [0,1]$, $\eta \in [0,\xi]$,

$$|(\lambda I - A)^{-1} v|_\xi \le C(1+|\lambda|)^{\frac{\xi-\eta}{2}-1} |v|_\eta \quad \text{for all } v \in \dot{\mathcal{C}}^\eta.$$

So the proof is complete. □

3 RESOLVENT ESTIMATES OF THE OPERATOR A_h

In this section, we present sectorial resolvent estimates for the finite element operator A_h. These estimates will look similarly to those involving the operator A. All of them will hold uniformly with respect to $h \in (0, h_0]$.

We shall start with recalling some facts used below and connected with application of the finite element method. First of all we state the well-known inverse property (cf. Ciarlet [11]), for $j = 0, 1$ and $1 \leq q \leq p \leq \infty$,

$$\|\chi\|_{p;j;\tau} \leq Ch^{-j-d(\frac{1}{q}-\frac{1}{p})}\|\chi\|_{q;0;\tau} \quad \text{for } h \in (0, h_0],\ \tau \in \mathcal{T}_h,\ \text{and } \chi \in S_h. \quad (3.1)$$

Let further P_h be the orthogonal (in the sense of L_2) projection onto S_h for which, given a function v,

$$(P_h v, \chi) = (v, \chi) \quad \text{for all } \chi \in S_h.$$

For our subsequent purposes we state the uniform boundedness of P_h in L_p norm (cf. Descloux [15] and Douglas, Dupont & Wahlbin [16]):

$$\|P_h v\|_p \leq C\|v\|_{p;\Omega_h} \leq C\|v\|_p \quad \text{for all } 1 \leq p \leq \infty \text{ and } v \in L_p. \quad (3.2)$$

Note that using (3.1), (3.2) with $p = \infty$, and well-known properties of the standard Lagrange interpolant I_h yields (cf. [7])

$$|P_h v|_1 \leq C|v|_{1;\Omega_h} \leq C|v|_1 \quad \text{for } v \in \dot{W}^1_\infty. \quad (3.3)$$

Moreover, an interpolation argument (see, e.g., Triebel [35, Theorem 1.3.3]), applied to (3.3) and (3.2) with $p = \infty$ (the last one clearly holds for all $v \in \dot{\mathcal{C}}^0$), implies for any fixed $\xi \in [0, 1)$, with the aid of (2.1) (for $\xi = 0$ this is in fact a direct consequence of (3.2) with $p = \infty$),

$$|P_h v|_\xi \leq C|v|_\xi \quad \text{for } v \in \dot{\mathcal{C}}^\xi. \quad (3.4)$$

Note in passing that (3.3) is an extension of (3.4) to the case $v \in W^1_\infty$.

Let next R_h be the standard Ritz projection onto S_h for which, given a function v,

$$(\nabla R_h v, \nabla\chi) = (\nabla v, \nabla\chi) \quad \text{for all } \chi \in S_h.$$

In what follows we shall use the following stability estimate for R_h in W^1_p norm, for all $1 \leq p \leq \infty$ and $v \in \dot{W}^1_\infty$,

$$\|R_h v\|_{p;1} \leq C\|v\|_{p;1;\Omega_h} + Ch|v|_{1;\Omega\setminus\Omega_h}. \quad (3.5)$$

For a proof of this in the case $p = \infty$, see [7]. Note however that the techniques in [7] can be slightly modified (using also the ideas developed in [27]) in order to obtain (3.5) as stated for $2 \leq p \leq \infty$.

Applying now (3.5) with $p = \infty$ to $(I_h v - v)$, we find for all $v \in \dot{W}^2_\infty$,

$$\begin{aligned} |R_h v - v|_{1;\Omega_h} &\leq |R_h(v - I_h v)|_1 + |(I_h v - v)|_{1;\Omega_h} \\ &\leq C|I_h v - v|_{1;\Omega_h} + Ch|v|_{1;\Omega\setminus\Omega_h} \leq Ch|v|_2. \end{aligned} \quad (3.6)$$

Using the fact that

$$P_h v - R_h v = P_h(v - R_h v), \quad (3.7)$$

and taking (3.3) into account, this yields in turn

$$|P_h v - R_h v|_1 \leq C|R_h v - v|_{1;\Omega_h} \leq Ch|v|_2 \quad \text{for } v \in \dot{W}^2_\infty. \quad (3.8)$$

Another estimate for $P_h - R_h$ will be contained in the following assertion:

Lemma 3.1. *For any fixed $1 < p < \infty$, we have*

$$\|(P_h - R_h)A^{-1}v\|_p \le Ch^2\|v\|_p \quad \textit{for } v \in L_p. \tag{3.9}$$

Proof. Let throughout the proof p and q be fixed, subject to $1 < p, q < \infty$.

Applied to $(I - I_h)w$, (3.5) yields

$$\|R_h(I - I_h)w\|_{q;1;\Omega_h} \le C\|(I - I_h)w\|_{q;1;\Omega_h} + Ch|w|_{1;\Omega\setminus\Omega_h} \quad \text{for } w \in \dot{W}^1_\infty.$$

With the aid of this estimate we obtain, assuming that q be sufficiently large and applying the inequality $|w|_1 \le C\|w\|_{q;2}$,

$$\|\nabla R_h(I - I_h)w\|_{q;\Omega_h} \le Ch\|w\|_{q;2} \quad \text{for } w \in \dot{W}^2_q.$$

Using this and the above argument, we get by elliptic regularity, with q sufficiently large,

$$\begin{aligned} \|\nabla(R_h - I)A^{-1}\psi\|_{q;\Omega_h} &= \|\nabla(R_h - I)\,(I - I_h)A^{-1}\psi\|_{q;\Omega_h} \\ &\le Ch\|A^{-1}\psi\|_{q;2} \le Ch\|\psi\|_q. \end{aligned} \tag{3.10}$$

Now, in order to apply a duality argument, we shall be based on the following identity, considered for suitable pairs of v and ψ,

$$\begin{aligned} \big((R_h - I)A^{-1}v, \psi\big) &= \big(v, (R_h - I)A^{-1}\psi\big) \\ &= \big(\nabla(I - I_h)A^{-1}v, \nabla(R_h - I)A^{-1}\psi\big) \\ &= \big(\nabla(I - I_h)A^{-1}v, \nabla(R_h - I)A^{-1}\psi\big)_{\Omega_h} \\ &\quad - \big(\nabla A^{-1}v, \nabla A^{-1}\psi\big)_{\Omega\setminus\Omega_h}. \end{aligned} \tag{3.11}$$

Using now (3.10) and the above reasonings, this leads in a standard way to the estimate

$$\|(R_h - I)A^{-1}v\|_p \le Ch^2\|v\|_p, \tag{3.12}$$

which will be therefore valid at least for p sufficiently close to 1 (the last term on the right-hand side of (3.11) is estimated as desired in the same way as in the proof of Lemma A.4 in Lin, Thomée & Wahlbin [20]). With the aid of the equality in the first line of (3.11), we conclude, by duality, that (3.12) holds as well for all p sufficiently large and further it is easily seen, by interpolation, that (3.12) is valid in fact for all $1 < p < \infty$.

It thus remains to combine (3.7), (3.2), and (3.12). □

We proceed now to showing resolvent estimates for the operator A_h. The first two results presented below are in fact found in [7]. We state however them here in view of their significance and for subsequent reference. For the second assertion we also propose a new proof, which is given, from our point of view, in a more straightforward way than in [7] and is based on using (2.9) with $\alpha = 1$. The latter fact is of particular interest because (2.9) with $\alpha = 1$ will be an essential ingredient in showing resolvent estimates of the operator A_h in Hölder norms.

Theorem 3.1. *For any fixed $\varphi \in (0, \frac{\pi}{2})$, we have*

$$|(\lambda I - A_h)^{-1}\chi|_0 \leq C(1+|\lambda|)^{-1}|\chi|_0 \quad \textit{for all } \lambda \notin \textit{Int } \Sigma_\varphi \textit{ and } \chi \in S_h. \tag{3.13}$$

Theorem 3.2. *For any fixed $\varphi \in (0, \frac{\pi}{2})$, we have*

$$\begin{gathered} |A_h(\lambda I - A_h)^{-1}\chi|_0 \leq C(1+|\lambda|)^{-\frac{1}{2}}|\chi|_1, \\ \textit{for all } \lambda \notin \textit{Int } \Sigma_\varphi \textit{ and } \chi \in S_h. \end{gathered} \tag{3.14}$$

The proof of (3.14) is given in fact only for $|\lambda| \leq \omega_0 h^{-2}$, with $\omega_0 > 0$ sufficiently small, because just this case needs to be settled. It was remarked in [7] that, in the opposite case $|\lambda| > \omega_0 h^{-2}$, (3.14) obtains in a trivial way. We shall therefore emphasize below on the case $|\lambda| \leq \omega_0 h^{-2}$, with $\omega_0 > 0$ sufficiently small.

Proof of Theorem 3.2 for $|\lambda| \leq \omega_0 h^{-2}$. We use the identity, for $\chi \in S_h$,

$$\begin{aligned} A_h(\lambda I - A_h)^{-1}\chi &= P_h A(\lambda I - A)^{-1}\chi \\ &\quad + \lambda A_h(\lambda I - A_h)^{-1}(P_h - R_h)\,(\lambda I - A)^{-1}\chi, \end{aligned} \tag{3.15}$$

both sides of which are well defined for all $\lambda \notin \text{Int } \Sigma_\varphi$. Applying (3.2) with $p = \infty$ and (2.7), we find

$$|P_h A(\lambda I - A)^{-1}\chi|_0 \leq C(1+|\lambda|)^{-\frac{1}{2}}|\chi|_1. \tag{3.16}$$

Next, by (3.8) and (2.9) with $\alpha = 1$, we get

$$|(P_h - R_h)\,(\lambda I - A)^{-1}\chi|_1 \leq Ch|(\lambda I - A)^{-1}\chi|_2 \leq Ch(1+|\lambda|)^{-\frac{1}{2}}|\chi|_1. \tag{3.17}$$

Now, given a linear bounded operator $B_h : S_h \to S_h$, denote $\|\!|B_h|\!\|_{S_h;1\to 0} := \sup_{\chi \in S_h}(|B_h\chi|_0/|\chi|_1)$. Inserting (3.17) and (3.16) into (3.15), we find

$$\begin{aligned} \|\!|A_h(\lambda I - A_h)^{-1}|\!\|_{S_h;1\to 0} &\leq C(1+|\lambda|)^{-\frac{1}{2}} \\ &\quad + C_1 h(1+|\lambda|)^{\frac{1}{2}}\|\!|A_h(\lambda I - A_h)^{-1}|\!\|_{S_h;1\to 0}. \end{aligned}$$

Clearly, if $C_1 h(1+|\lambda|)^{\frac{1}{2}} \leq \frac{1}{2}$, this implies

$$\|\!|A_h(\lambda I - A_h)^{-1}|\!\|_{S_h;1\to 0} \leq C(1+|\lambda|)^{-\frac{1}{2}},$$

so that the result follows for $|\lambda| \leq \omega_0 h^{-2}$, if ω_0 is sufficiently small. □

The above estimate (3.13) implies similar estimates for the whole Lebesgue scale L_p, $1 \leq p \leq \infty$.

Theorem 3.3. *For any fixed $\varphi \in (0, \frac{\pi}{2})$, we have*

$$\begin{gathered} \|(\lambda I - A_h)^{-1}\chi\|_p \leq C(1+|\lambda|)^{-1}\|\chi\|_p, \\ \textit{for all } 1 \leq p \leq \infty,\ \lambda \notin \textit{Int } \Sigma_\varphi,\ \textit{and } \chi \in S_h. \end{gathered} \tag{3.18}$$

Proof. Immediate by duality and interpolation. □

Our next result gives a resolvent estimate of A_h in W^1_∞ norm. Note that the below proof of this assertion will be essentially based on the use of the estimate (2.9) with $\alpha = 1$.

Theorem 3.4. *For any fixed $\varphi \in (0, \frac{\pi}{2})$, we have*

$$|(\lambda I - A_h)^{-1}\chi|_1 \le C(1+|\lambda|)^{-1}|\chi|_1 \quad \textit{for all } \lambda \notin \textit{Int } \Sigma_\varphi \textit{ and } \chi \in S_h. \tag{3.19}$$

Proof. We use the identity (cf. [7]), for $\chi \in S_h$,

$$(\lambda I - A_h)^{-1}\chi = P_h(\lambda I - A)^{-1}\chi + A_h(\lambda I - A_h)^{-1}(P_h - R_h)\,(\lambda I - A)^{-1}\chi, \tag{3.20}$$

where both sides are well defined for $\lambda \notin \text{Int } \Sigma_\varphi$. By (3.3) and (2.9) with $\alpha = 0$, we have

$$|P_h(\lambda I - A)^{-1}\chi|_1 \le C(1+|\lambda|)^{-1}|\chi|_1. \tag{3.21}$$

Next, denoting

$$G := A_h(\lambda I - A_h)^{-1}(P_h - R_h)\,(\lambda I - A)^{-1}\chi, \tag{3.22}$$

we find by (3.1), (3.14), (3.8), and (2.9) with $\alpha = 1$,

$$\begin{aligned} |G|_1 &\le Ch^{-1}|G|_0 \\ &\le Ch^{-1}(1+|\lambda|)^{-\frac{1}{2}}|(P_h - R_h)\,(\lambda I - A)^{-1}\chi|_1 \\ &\le C(1+|\lambda|)^{-\frac{1}{2}}|(\lambda I - A)^{-1}\chi|_2 \le C(1+|\lambda|)^{-1}|\chi|_1. \end{aligned} \tag{3.23}$$

The claim thus follows by combining (3.20), (3.21), and (3.23). □

Now we can show estimates in pairs of Hölder norms.

Theorem 3.5. *For any fixed $\varphi \in (0, \frac{\pi}{2})$, $\xi \in [0,1]$, and $\eta \in [0, \xi]$, we have*

$$\begin{aligned} &|(\lambda I - A_h)^{-1}\chi|_\xi \le C(1+|\lambda|)^{\frac{\xi-\eta}{2}-1}|\chi|_\eta, \\ &\quad \textit{for all } \lambda \notin \textit{Int } \Sigma_\varphi \textit{ and } \chi \in S_h. \end{aligned} \tag{3.24}$$

Proof. First of all, by (3.3), (3.5)with $p = \infty$, and (2.31) with $\xi = 1$, we have

$$\begin{aligned} |(P_h - R_h)\,(\lambda I - A)^{-1}\chi|_1 &\le C|(\lambda I - A)^{-1}\chi|_1 \\ &\le C(1+|\lambda|)^{-\frac{\eta+1}{2}}|\chi|_\eta. \end{aligned} \tag{3.25}$$

With G defined as above (see (3.22)), using further (2.12) with A_h substituted for A, (3.19), and (3.25) yields

$$\begin{aligned} |G|_1 &\le |(P_h - R_h)\,(\lambda I - A)^{-1}\chi|_1 \\ &\quad + |\lambda(\lambda I - A_h)^{-1}(P_h - R_h)\,(\lambda I - A)^{-1}\chi|_1 \\ &\le C|(P_h - R_h)\,(\lambda I - A)^{-1}\chi|_1 \\ &\le C(1+|\lambda|)^{-\frac{\eta+1}{2}}|\chi|_\eta. \end{aligned} \tag{3.26}$$

Combining now (3.4) (if $\xi = 1$, we use instead (3.3)) and (2.31), this yields

$$|P_h(\lambda I - A)^{-1}\chi|_\xi \leq C|(\lambda I - A)^{-1}\chi|_\xi \leq C(1+|\lambda|)^{\frac{\xi-\eta}{2}-1}|\chi|_\eta. \tag{3.27}$$

At the same time, applying (3.14) and (3.25) allows one to get as well

$$\begin{aligned} |G|_0 &\leq C(1+|\lambda|)^{-\frac{1}{2}}|(P_h - R_h)\,(\lambda I - A)^{-1}\chi|_1 \\ &\leq C(1+|\lambda|)^{-\frac{\eta}{2}-1}|\chi|_\eta. \end{aligned} \tag{3.28}$$

It follows from (3.26) and (3.28) that

$$|G|_\xi \leq C|G|_1^{\xi}|G|_0^{1-\xi} \leq C(1+|\lambda|)^{\frac{\xi-\eta}{2}-1}|\chi|_\eta. \tag{3.29}$$

Finally, the claim obtains by combining (3.20), (3.27), and (3.29). □

We shall also derive a finite element analogue of Theorem 2.7 involving (Hölder, Lebesgue) pairs.

Theorem 3.6. *For any fixed $\varphi \in (0, \frac{\pi}{2})$, $\xi \in [0,1]$, and $q \in \left(\frac{d}{2-\xi}, \infty\right]$, we have*

$$\begin{aligned} |(\lambda I - A_h)^{-1}\chi|_\xi \leq C(1+|\lambda|)^{\frac{\xi}{2}+\frac{d}{2q}-1}\|\chi\|_q, \\ \textit{for all } \lambda \notin \textit{Int}\,\Sigma_\varphi \textit{ and } \chi \in S_h. \end{aligned} \tag{3.30}$$

Proof. Let throughout the proof G be just the same as in (3.22).

Assume first that $q \in (d, \infty]$. By (3.25) and (2.22) with $\xi = 1$, we obtain

$$|(P_h - R_h)\,(\lambda I - A)^{-1}\chi|_1 \leq C(1+|\lambda|)^{\frac{d}{2q}-\frac{1}{2}}\|\chi\|_q, \tag{3.31}$$

and a similar argument shows that

$$|P_h(\lambda I - A)^{-1}\chi|_1 \leq C(1+|\lambda|)^{\frac{d}{2q}-\frac{1}{2}}\|\chi\|_q. \tag{3.32}$$

Inserting now (3.31) into (3.26), we find

$$|G|_1 \leq C(1+|\lambda|)^{\frac{d}{2q}-\frac{1}{2}}\|\chi\|_q. \tag{3.33}$$

So combining (3.20), (3.32), and (3.33) shows the claim at least for $\xi = 1$.

Let further $\xi = 0$ and $q \in \left(\frac{d}{2}, \infty\right]$. From now on we assume that $d \geq 2$ (the below reasonings can be easily modified to the case $d = 1$). If, additionally, $q = \infty$, the result directly obtains by (3.13). We may thus consider only the case $q \in \left(\frac{d}{2}, \infty\right)$. Let p be a fixed number such that $p > q$ and $p \in (d, \infty)$. Clearly, p can be chosen such that $\frac{1}{q} - \frac{1}{p} < \frac{1}{d}$. Then, using the inverse inequality $|A_h\chi|_0 \leq Ch^{-1}|\chi|_1$, we get, since $p > d$ and the result is already proved for $\xi = 1$,

$$\begin{aligned} |G|_0 &\leq Ch^{-1}|(\lambda I - A_h)^{-1}(P_h - R_h)\,(\lambda I - A)^{-1}\chi|_1 \\ &\leq Ch^{-1}(1+|\lambda|)^{\frac{d}{2p}-\frac{1}{2}}\|(P_h - R_h)\,(\lambda I - A)^{-1}\chi\|_p. \end{aligned}$$

Combining further this with (3.9), (2.12), and (2.6) yields

$$|G|_0 \le Ch(1+|\lambda|)^{\frac{d}{2p}-\frac{1}{2}}\|\chi\|_p.$$

Defining now $r \in [1,p)$ by $\frac{1}{r} := \frac{1}{p} + \frac{1}{d}$ and using the inverse inequality (3.1) with $j=0$, it follows from the last estimate that

$$|G|_0 \le C(1+|\lambda|)^{\frac{d}{2p}-\frac{1}{2}}\|\chi\|_r. \tag{3.34}$$

At the same time, as a direct consequence of (3.29) with $\xi = \eta = 0$, we have

$$|G|_0 \le C(1+|\lambda|)^{-1}\|\chi\|_\infty. \tag{3.35}$$

An interpolation argument, applied to (3.34) and (3.35), thus leads to the estimate

$$|G|_0 \le C(1+|\lambda|)^{\frac{d}{2q}-1}\|\chi\|_q. \tag{3.36}$$

Moreover, by (3.2) and (2.19), both with $p=\infty$, we have

$$|P_h(\lambda I - A)^{-1}\chi|_0 \le C|(\lambda I - A)^{-1}\chi|_0 \le C(1+|\lambda|)^{\frac{d}{2q}-1}\|\chi\|_q. \tag{3.37}$$

Altogether (3.20), (3.36) and (3.37) show the claim for $\xi = 0$.

It remains to get the result in the case $\xi \in (0,1)$. In order to do this, note that we can write any fixed $q \in \left(\frac{d}{2-\xi}, \infty\right]$ in the form

$$\frac{1}{q} = \frac{2-\xi}{d} - \varepsilon, \quad \varepsilon > 0.$$

Assuming that $\varepsilon < \frac{1}{d}$ and defining then q_0 and q_1 by

$$\frac{1}{q_0} := \frac{2}{d} - \varepsilon, \quad \frac{1}{q_1} := \frac{1}{d} - \varepsilon,$$

we have by (3.30) with $\xi = 1$ and $\xi = 0$ (already shown) and by (3.2),

$$\begin{aligned} |(\lambda I - A_h)^{-1}P_h v|_1 &\le C(1+|\lambda|)^{\frac{d}{2q_1}-\frac{1}{2}}\|P_h v\|_{q_1} \\ &\le C(1+|\lambda|)^{\frac{d}{2q_1}-\frac{1}{2}}\|v\|_{q_1}, \end{aligned}$$

and

$$|(\lambda I - A_h)^{-1}P_h v|_0 \le C(1+|\lambda|)^{\frac{d}{2q_0}-1}\|v\|_{q_0}.$$

Using these estimates, we find by interpolation

$$|(\lambda I - A_h)^{-1}P_h v|_\xi \le C(1+|\lambda|)^{\frac{\xi}{2}+\frac{d}{2q}-1}\|v\|_q,$$

whence the result with $0 < \xi < 1$ follows by taking $v = \chi$.

If the above ε does not satisfy the restriction $\varepsilon < \frac{1}{d}$, this situation can be considered after all by using an interpolation argument (as it is done in the proof of Theorem 2.6).

So the proof is complete. □

In the case of purely Lebesgue pairs we have the following result:

Theorem 3.7. *For any fixed* $\varphi \in (0, \frac{\pi}{2})$ *and* $1 \le q \le p \le \infty$ *such that* $\frac{1}{q} - \frac{1}{p} < \frac{2}{d}$, *we have*

$$\|(\lambda I - A_h)^{-1}\chi\|_p \le C(1+|\lambda|)^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-1}\|\chi\|_q,$$
$$\text{for all } \lambda \notin \mathit{Int}\,\Sigma_\varphi \text{ and } \chi \in S_h. \tag{3.38}$$

Proof. In the case $p = \infty$, $\frac{1}{q} < \frac{2}{d}$, the result obtains straightforwardly by (3.30) with $\xi = 0$. By duality, the claim also follows for $1 - \frac{1}{p} < \frac{2}{d}$, $q = 1$. Finally, the general case can be then shown by interpolation. □

The following results unite and generalize in some sense the assertions of Theorems 3.6 and 3.7.

Theorem 3.8. *Let* $\xi \in [0,1]$, $q \in [1,\infty]$, *and* $m \in \mathbb{N} \cup \{0\}$ *be such that*

$$\frac{1}{q} < \frac{2m+2-\xi}{d}.$$

Then, for any fixed $\varphi \in (0, \frac{\pi}{2})$, *we have*

$$|(\lambda I - A_h)^{-(m+1)}\chi|_\xi \le C(1+|\lambda|)^{\frac{\xi}{2}+\frac{d}{2q}-(m+1)}\|\chi\|_q,$$
$$\text{for all } \lambda \notin \mathit{Int}\,\Sigma_\varphi \text{ and } \chi \in S_h.$$

Theorem 3.9. *Let* $1 \le q \le p \le \infty$ *and* $m \in \mathbb{N} \cup \{0\}$ *be such that*

$$\frac{1}{q} - \frac{1}{p} < \frac{2m}{d}.$$

Then, for any fixed $\varphi \in (0, \frac{\pi}{2})$, *we have*

$$\|(\lambda I - A_h)^{-m}\chi\|_p \le C(1+|\lambda|)^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-m}\|\chi\|_q,$$
$$\text{for all } \lambda \notin \mathit{Int}\,\Sigma_\varphi \text{ and } \chi \in S_h.$$

The proofs of these results are based on an induction argument and are similar to their continuous analogues (see the proofs of Theorems 2.10 and 2.11).

In conclusion we note that using the above results, combined with more or less standard reasonings, allows one to obtain finite element analogues of Theorems 2.8, 2.9, and 2.12. We omit the corresponding statements.

References

[1] Sh.Agmon, On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems, Comm. Pure Appl. Math. **15**(1962), 119-147.

[2] Sh.Agmon, A.Douglis, and L.Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, Comm. Pure Appl. Math. **12**(1959), 623-727.

[3] N.Yu.Bakaev, Stability estimates for a general discretization method, Dokl. Akad. Nauk SSSR **309**(1989), 11-15 (in Russian); English transl. in Soviet Math. Dokl. **40**(1990).

[4] N.Yu.Bakaev, Maximum norm resolvent estimates for elliptic finite element operators, BIT **41**(2001), 215-239.

[5] N.Yu.Bakaev, Analysis of fully discrete approximations to parabolic problems with rough or distribution-valued initial data, to appear in Numer. Math.

[6] N.Yu.Bakaev, S.Larsson, and V.Thomée, Long time behavior of backward difference type methods for parabolic equations with memory in Banach space, East-West J. Numer. Math. **6**(1998), 185-206.

[7] N.Yu.Bakaev, V.Thomée, and L.B.Wahlbin, Maximum-norm estimates for resolvents of elliptic finite element operators, to appear in Math. Comp.

[8] S.Campanato, Generation of analytic semigroups by elliptic operators of second order in Hölder spaces, Ann. Sc. Norm. Sup. Pisa **8**(1981), 495-512.

[9] P.Cannarsa, B.Terreni, and V.Vespri, Analytic semigroups generated by non-variational elliptic systems of second order under Dirichlet boundary conditions, J. Math. Anal. Appl. **112**(1985), 56-103.

[10] H.Chen, An L^2 and L^∞-Error Analysis for Parabolic Finite Element Equations with Applications by Superconvergence and Error Expansions, Thesis, Heidelberg University, 1993.

[11] Ph.Ciarlet, The Finite Element Method for Elliptic Problems, North Holland Publishing Company, Amsterdam - New York - Oxford, 1978.

[12] F.Colombo and V.Vespri, Generation of analytic semigroups in $W^{k,p}(\Omega)$ and $C^k(\overline{\Omega})$, Diff. Int. Eqs **9**(1996), 421-436.

[13] M.Crouzeix, S.Larsson, and V.Thomée, Resolvent estimates for elliptic finite element operators in one dimension, Math. Comp. **63**(1994), 121-140.

[14] M.Crouzeix and V.Thomée, Resolvent estimates in l_p for discrete Laplacians on irregular meshes and maximum-norm stability of parabolic finite difference schemes, Comput. Meth. Appl. Math. **1**(2001), 3-17.

[15] J.Descloux, On finite element matrices, SIAM J. Numer. Anal. **9**(1972), 260-265.

[16] J.Douglas, Jr., T.Dupont, and L.B.Wahlbin, The stability in L^q of the L^2-projection into finite-element function spaces, Numer. Math. **23**(1975), 193-197.

[17] S.D.Èĭdel'man and S.D.Ivasišen, Investigation of the Green's matrix of a homogeneous parabolic boundary value problem, Trans. Moscow Math. Soc. **23**(1970), 179-242.

[18] H.Fujii, Some remarks on finite element analysis of time-dependent field problems, in Theory and Practice in Finite Element Structural Analysis, University of Tokyo Press (1973), 91-106.

[19] P.Grisvard, Équations différentielles abstraites, Ann. Sci. Éc. Norm. Super. **4**(1969), 311-395.

[20] Y.Lin, V.Thomée, and L.Wahlbin, Ritz-Volterra projections to finite-element spaces and applications to integrodifferential and related equations, SIAM J. Numer. Anal. **28**(1991), 1047-1070.

[21] A.Lunardi, Interpolation spaces between domains of elliptic operators and spaces of continuous functions with applications to nonlinear parabolic equations, Math. Nachr. **121**(1985), 295-318.

[22] A.Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Basel - Boston - Berlin, 1995.

[23] X.Mora, Semilinear parabolic problems define semiflow on C^k spaces, Trans. Amer. Math. Soc. **278**(1983), 21-55.

[24] J.A.Nitsche and M.F.Wheleer, L_∞-boundedness of the finite element Galerkin operator for parabolic problems, Numer. Funct. Anal. Optimization **4**(1981-1982), 325-353.

[25] C.Palencia, Maximum norm analysis of completely discrete finite element methods, SIAM J. Numer. Anal. **33**(1996), 1654-1658.

[26] R.Rannacher, L^∞-stability estimates and asymptotic error expansion for parabolic finite element equations, in Extrapolation and Defect Correction, Bonner Math. Schriften **228**(1991), 74-94.

[27] R.Rannacher and R.Scott, Some optimal error estimates for piecewise linear finite element approximations, Math. Comp. **38**(1982), 437-445.

[28] A.H.Schatz, V.Thomée, and L.B.Wahlbin, Maximum norm stability and error estimates in parabolic finite element equations, Comm. Pure Appl. Math. **33**(1980), 265-304.

[29] A.H.Schatz, V.Thomée, and L.B.Wahlbin, Stability, analyticity, and almost best approximation in maximum norm for parabolic finite element equations, Comm. Pure Appl. Math. **51**(1998), 1349-1385.

[30] A.H.Schatz and L.B.Wahlbin, On the quasi-optimality in L_∞ of the $\mathring{H}^1$-projection into finite element spaces, Math. Comp. **38**(1982), 1-22.

[31] H.B.Stewart, Generation of analytic semigroups by strongly elliptic operators, Trans. Amer. Math. Soc. **199**(1974), 141-161.

[32] V.Thomée, Galerkin Finite Element Methods for Parabolic Problems, Springer Series in Computational Mathematics, Springer - Verlag, Berlin - Heidelberg - New York, 1997.

[33] V.Thomée and L.B.Wahlbin, Maximum-norm stability and error estimates in Galerkin methods for parabolic equations in one space variable, Numer. Math. **41**(1983), 345-371.

[34] V.Thomée and L.B.Wahlbin, Stability and analyticity in maximum-norm for simplicial Lagrange finite element semidiscretizations of parabolic equations with Dirichlet boundary conditions, Numer. Math. **87**(2000), 373-389.

[35] H.Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdam - New York - Oxford, 1978.

[36] L.B.Wahlbin, A quasioptimal estimate in piecewise polynomial Galerkin approximation of parabolic problems, in Numerical Analysis (Dundee, 1981), Lecture Notes in Mathematics, No. 912, Springer-Verlag, Berlin - New York, 1982, 230-245.

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