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Helsinki University of Technology
Department of Engineering Physics and Mathematics
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Abstract: *In this thesis we study quenching and blowup problems for reaction diffusion equations with Cauchy-Dirichlet data. We give sufficient conditions for certain reaction terms under which quenching or blowup can occur. Furthermore we show that the set of quenching points is finite for certain nonlinearities. The main results concern the asymptotic behavior of the solution in a neighborhood of a quenching or blowup point. We prove two kinds of asymptotic theorems. First we study quenching or blowup rate results and then give precise asymptotic expressions for solutions in a backward space-time parabola near a quenching point for certain reaction terms.*

AMS subject classifications: 35K20, 35K55, 35K57, 35K60, 35B05, 35B40

Keywords: Reaction-diffusion equation, quenching, quenching set, quenching rate, asymptotic behavior of solutions, refined asymptotics, blow-up, blow-up set, blow-up rate

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Timo Salin

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Abstract: In this work we study reaction diffusion equations, i.e., $u_t - \Delta u = f(u, \nabla u; x, t)$, where the solution $u = u(x, t)$ is real valued and defined for $(x, t) \in \Omega \times [0, T]$, $0 < T \leq \infty$ and $\Omega \subset R^N$. The term Δu is the diffusion term, and $f(u, \nabla u; x, t)$ is the reaction term. We take Ω to be a bounded subset of R^N and assume Cauchy-Dirichlet data, i.e., u is given on the boundary $\partial\Omega$ and at the initial time $t = 0$.

These equations have many application in physics, chemistry or biology. For example, chemical reactions, population dynamics or a theory of combustion are modelled by reaction diffusion equations. In particular we are interested in equations where f is in some sense singular with respect to u. These situations consist roughly speaking of two categories.

In the first case $f \to \infty$ as $u \to \infty$, for example $f(u) = e^u$ or $f(u, u_x) = u^p - u_x^2$. These applications arises in the combustion theory, in population genetics or in population dynamics. The main interest is a possibility that there are solutions which can tend to infinity in finite time. This phenomenon is called blowup.

In another case we have reaction terms that satisfy $f \to \infty$ as $u \to K$ for some $K \in [0, \infty)$, for example $f(u) = -\frac{1}{u}$. This type of reaction diffusion equations with singular reaction term arises in the study of electric current transients in polarized ionic conductors. The problem can also be considered as a limiting case of models in chemical catalyst kinetics (Langmuir-Hinshelwood model) or of models in enzyme kinetics. The equation has been extensively studied under assumptions implying that the solution $u(x, t)$ approaches K in finite time. The reaction term then tends to infinity and the smooth solution ceases to exist. This phenomenon is called quenching or extinction.

Our contribution consists of giving sufficient conditions for certain weakly singular reaction terms under which quenching or blowup can occur. Furthermore we show that the set of quenching points is finite for certain nonlinearities. The main results concern the asymptotic behavior of the solution in a neighborhood of a quenching or blowup point. We prove two kinds of asymptotic theorems. First we study quenching or blowup rate results and then give precise asymptotic expressions for solutions in a backward space-time parabola near a quenching point for certain reaction terms.

Keywords: Reaction-diffusion equation, quenching, quenching set, quenching rate, asymptotic behavior of solutions, refined asymptotics, blow-up, blow-up set, blow-up rate.

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[III] T. Salin, *On a refined asymptotic analysis for the quenching problem*, Helsinki Univ. Techn. Inst. Math. Research Report A457. 2003.

[IV] T. Salin, *The quenching problem for the N-dimensional ball*, Helsinki Univ. Techn. Inst. Math. Research Report A459. 2003.

1 Introduction

1.1 Reaction diffusion equations

By a reaction diffusion equation we mean an equation of the form

$$u_t - \Delta u = f(u, \nabla u; x, t), \tag{1.1}$$

where the solution $u = u(x,t)$ is real valued and defined for $(x,t) \in \Omega \times [0,T]$, $0 < T \leq \infty$ and $\Omega \subset R^N$. The term Δu is the diffusion term, and $f(u, \nabla u; x, t)$ is the reaction term. More generally, the diffusion term may be of type $\mathcal{A}(u)$, where $\mathcal{A}$ is a second-order elliptic operator, which may be nonlinear and degenerate. In this work, however, we are only interested in the case where the diffusion term equals the Laplacian. We take Ω to be a bounded subset of R^N and assume Cauchy-Dirichlet data, i.e., u is given on the boundary $\partial\Omega$ and at the initial time $t = 0$.

Of primary interest to us are reaction terms $f = f(u)$, i.e., terms not explicitly depending on ∇u, x or t. Write, formally, $e^{t\Delta}$ to denote the semigroup generated by the operator Δ with Dirichlet boundary conditions in a certain function space. Then the variation of constants formula for the equation (1.1) is

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f(u(s))ds. \tag{1.2}$$

A method to prove local existence and uniqueness for the equation (1.1) is to use the contraction mapping principle in (1.2). The crucial property on f is then that f be locally Lipschitz continuous.

This solution may be locally continued. In some cases, the solution exists for all subsequent time (global existence). However, for certain f and u_0 there is a time $T < \infty$ such that $\|u(t)\|_\infty \to \infty$, as $t \uparrow T$. This phenomenon is called blowup.

In the simple case with $f = 0$ in (1.1), the equation (1.1) is the (linear) diffusion or heat equation. Take, for example, $u = 0$ on the boundary and $u_0 \in C(\overline{\Omega})$. Then we can write the solution in the closed form $u(t) = e^{t\Delta}u_0$. From this expression we can easily verify several qualitative properties of $u(t)$. In particular we observe that $u(t)$ exists globally and no blowup can occur.

This example obviously tells us that a (possible) blowup in the equation (1.1) is a consequence of the cumulative effect of the nonlinearity $f(u)$. Actually this is elementary for ordinary differential equations. Namely, if we set $\Delta u = 0$ and $f(u) = u^p$ with $p > 1$ in (1.1), and study the ODE:

$$u' = u^p, \quad t > 0; \qquad u(0) = 1, \tag{1.3}$$

we get $u(t) = \big[1 + (1-p)t\big]^{\frac{1}{1-p}}$. Thus the solution is smooth for $t \in (0, \frac{1}{p-1})$ and $u(t) \to \infty$ as $t \uparrow \frac{1}{p-1}$, i.e., u blows up.

For partial differential equations, as (1.1), the situation is much more complicated. In general, we cannot solve the equations explicitly, and the possibility of blowup is therefore difficult to examine.

The first fundamental paper concerning the blowup problem for the reaction diffusion equation was written by Fujita [22]. He studied the Cauchy problem for the equation $u_t - \Delta u = u^{1+\alpha}$, $\alpha > 0$ and proved that if $0 < N\alpha < 2$ (N is the space dimension), then the initial value problem had no nontrivial global solutions while if $N\alpha > 2$, there were nontrivial global solutions. In this second case it was essential that the initial values were sufficiently small. After the publication of this paper the blowup phenomenon for the reaction diffusion equations has been the object of intensive research. See, for example, the review articles [25] and [44], and the references therein.

Another type of situation where the reaction diffusion equation does not have a global (smooth) solution are the equations in which the reaction term is in some sense singular for finite u. A typical example is (1.1) with $f(u) = -u^{-p}$, $p > 0$. In this case it is conceivable that there exists a time T such that $\inf_{x\in\Omega} u \downarrow 0$, as $t \uparrow T$. Then the reaction term blows up, and the smooth solution ceases to exist. This phenomenon is called quenching (or in some papers [25] extinction).

As in the case of the blowup problem, the quenching behavior is also caused by a nonlinear reaction term. We can conclude, by the parabolic Harnack's inequality, that quenching is impossible if, e.g., we have a uniformly elliptic operator as the diffusion term in (1.1) and $f \equiv 0$ with $u = 1$ on the boundary.

In the case of ordinary differential equations we can demonstrate quenching by a simple example. We replace the term u^p in (1.3) by $-u^{-p}$, $p > 0$, and solve it to get $u(t) = \left[1-(1+p)t\right]^{\frac{1}{1+p}}$. From this we obtain that the solution is smooth for $t \in (0, \frac{1}{p+1})$ and $u(t) \to 0$ as $t \uparrow \frac{1}{p+1}$, i.e., u quenches. When we move on to study the quenching for the partial differential equation (1.1), we observe that the diffusion term Δu resists quenching, and the situation is consequently harder to analyze.

Although the blowup and the quenching problems somewhat resemble each other, a qualitative difference is that in the blowup problem the solution $u(t)$ becomes unbounded while in the quenching problem some derivative of the solution $u(t)$ blows up. Typically the time derivative u_t blows up in quenching problems, a fact which makes these equations challenging. The changes with respect to time happen faster and faster. Therefore (for example) the analysis by using the contraction mapping principle in (1.2) does not tell us much about the qualitative properties of the solution near the quenching point, because the size of the time steps tends to zero.

The original paper concerning the quenching problem was written by Kawarada [37]. This paper did initiate a wide study of the quenching problem by many authors, including work on existence and nonexistence, structure or size of quenching points, asymptotic behavior of the solutions in space and time near the quenching points etc.. In the next section we give an overview of the results. See also the review articles [25, 38, 43, 44].

In this work we concentrate on the quenching problem for equations of type (1.1). We also analyze the blowup problem for equations of type (1.1).

The equation (1.1) has many applications in physics, chemistry and biology (see [30, 34, 58]). Below we present some situations where blowup or quenching behavior are possible and which are essentially connected to this work.

(a) The theory of combustion and population genetics (see [25, 44, 55, 62] and references therein). There are two classical scalar models. One of them is the exponential reaction model where $f(u) = \delta e^u$ in (1.1). This model is important in combustion theory where it is also known as the Frank-Kamenetsky equation [62]. For instance, combustion of a one-dimensional solid fuel is described by the set of equations:

$$T_t = T_{xx} + \delta\varepsilon c \exp\left(\tfrac{T-1}{\varepsilon T}\right), \qquad c_t = -\varepsilon\Gamma\delta c \exp\left(\tfrac{T-1}{\varepsilon T}\right),$$

where T and c represent respectively the fuel temperature and concentration, and δ, Γ, ε are (positive) physical constants. Typically, ε represents the inverse of the activation energy. If we assume $0 < \varepsilon \ll 1$, and look for solutions in the form $T = 1 + \varepsilon u + ...$ and $c = 1 - \varepsilon C_1 + ...$, we are led to $u_t = u_{xx} + \delta e^u$ and $(C_i)_t = \Gamma\delta e^u$.

The other classical blowup equation is (1.1) with $f(u) = u^p$.

(b) Population dynamics (see [55]). In this case the equation (1.1) was first introduced with $f(u) = |u|^{p-1}u - b|\nabla u|^q$ $(p > 1, q \geq 1)$ (see [10]). In this model it was studied how the gradient damping term $b|\nabla u|^q$ affects the possible blowup behavior. The term $f_1 = |u|^{p-1}u$ describes the births and the term $-b|\nabla u|^q$ describes the deaths within a population. In particular, the dissipative gradient term represents the action of a predator which destroys the individuals during their displacements. The births can also be described by an exponential term, i.e., $f_1(u) = e^u$ or $f_1(u) = ue^u$.

(c) In connection with the diffusion equation generated by a polarization phenomena in ionic conductors (see [37] and references therein). In the paper [37] the equation (1.1) was studied in one space dimension with $f(u) = \frac{1}{1-u}$ and $u \equiv 0$ on the parabolic boundary. In this case quenching means that $u \uparrow 1$. Note that these equations are usually written in the form where the singularity appears at $u = 0$, i.e., $f(u) = -\frac{1}{u}$.

(d) As a limiting case of models in chemical catalyst kinetics (Langmuir-Hinshelwood model) or of models in enzyme kinetics (see [17, 49] and references therein): In this case $f = f(u, \varepsilon)$ is a smooth function for $\varepsilon > 0$, and $f(u, \varepsilon) \to f(u)$, as $\varepsilon \to 0$, where $f(u)$ is negative for $u > 0$ and singular at $u = 0$. Actually the reaction term is denoted by $f = f(u)\chi(\{u > 0\})$ to emphasize that the reaction ceases at $u = 0$.

(e) The problem of a superconducting vortex intersecting with the boundary between vacuum and a superconducting material (see [9, 47]). In the paper [47], a vortex line at time $t \geq 0$ is viewed as

$$L(t) = \{(x, y, z) = (x, 0, u(x, t)) | x \in \Omega\},$$

where $\Omega = (-1,1)$ or $\Omega = R$, and $u > 0$ is a regular function. The physical derivation gives that $u(x,t)$ satisfies (1.1) with $f(u) = e^{-u}H_0 - F_0(u)$, where H_0 is the applied magnetic field assumed to be constant, F_0 is a regular function satisfying $F_0(u) \sim \frac{1}{u}$ and $F_0'(u) \sim -\frac{1}{u^2}$ as $u \to 0$. In this model , a vortex reconnection with the boundary (the plane $z = 0$) means quenching.

(f) In connection with phase transitions, when we study the motion of the borderline between liquids and solids (see [16, 39]). In this application, also the diffusion term is nonlinear and the equation takes the form

$$u_t - \nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = -\frac{1}{u}.$$

(g) In connection with detonation theory (see [23] and references therein): In this case both the diffusion term and the reaction term are nonlinear. Thus,

$$u_t - \ln\left(\frac{e^{cuu_{xx}}-1}{cuu_{xx}}\right) = \ln(u) - \frac{1}{2}u_x^2$$

where c is a nondimensional positive constant representing the chemical properties. This equation was studied in [23] with Neumann boundary conditions.

In models (a) and (b) it is possible that u blows up in finite time. Correspondingly, in models (c)-(g) it is possible that u quenches in finite time.

We shall now briefly explain some general aspects of the equations (1.1) and comment on the literature concerning the subject.

The equation (1.1) has been studied extensively (see the books [12, 30, 34, 41, 45, 54, 58], and references therein). The results obtained concern existence, uniqueness, continuous dependence, stability, smoothness and asymptotics of solutions etc.

The geometric theory for the equations (1.1) has been handled in [34]. In this context, a basic approach is to write the partial differential equation as an ordinary differential equation in a Banach space (involving unbounded operators), and then try to extend the ideas and theorems from the theory of finite dimensional dynamical systems to this infinite dimensional setting.

This approach has led to the development of the theory of C_0-semigroups [29, 48]. The problem is to give necessary and sufficient conditions under which the problem is well posed. This means that the equation has a unique solution, which depends continuously on the initial function. As is well known, the problem (1.1) with $f = 0$ and $\mathcal{A}u$ as the (linear) diffusion term is well posed provided that $\mathcal{A}$ is the generator of a C_0-semigroup, see [29, 48].

In the case of linear equations, homogeneous or nonhomogenous, a semigroup approach gives the solution to the problem explicitly, while for nonlinear equations as described above, we need also fixed point theorems to settle the existence. For a detailed treatise on the basic theory of abstract parabolic equations in general Banach spaces, see [46]. Applications in [46] concern both linear and nonlinear equations. Basic results on fixed point theorems applied to partial differential equations of type (1.1) can also be found in [12, 45].

Because the equations (1.1) are of second order, we are able to apply the parabolic maximum principle, or some comparison methods that are essentially consequences of this principle, in the analysis. The techniques involving sub- and super solutions or invariant regions, also belong to this category. Roughly speaking these methods can be used in two ways.

In the first case we take existence for granted and derive a priori bounds for solutions. These bounds are useful, for example, in the study of regularity or asymptotics of solutions. A basic idea is to make an appropriate guess on the sign of a certain function P, where $P = P\big(u, u_x, u_t, f(u)\big)$ and u is a solution of (1.1). Then we deduce the corresponding parabolic inequality for P from (1.1), and use the maximum principle.

In the second case these methods can be used to prove existence results. A strategy in this case is to first find a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of a corresponding boundary value problem (1.1) such that $\underline{u} \leq \overline{u}$, and then use this to prove that there exists a solution satisfying $\underline{u} \leq u \leq \overline{u}$. See the books [12, 20, 30, 54, 58] for a comprehensive interpretation of the methods concerning the maximum principle and its applications.

1.2 Earlier results

Typical research subjects for blowup and quenching problems are:

(a) What are necessary and sufficient conditions for blowup or quenching?

(b) What can be said about the set of blowup- or quenching points? Can blowup or quenching take place on an entire interval, or is blowup or quenching possible only on distinct points?

(c) What kind of asymptotic behavior do solutions obey near the blowup or quenching points?

(d) What can be proved on solutions after blowup or quenching?

In the following we present known results for quenching and blowup problems, which are essentially connected to this work.

Consider the equation

$$\begin{gathered} u_t - \Delta u = f(u), \qquad x \in \Omega, \quad t \in (0,T), \\ u(x,t) = 1, \quad x \in \partial\Omega, \quad t \in (0,T), \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{1.4}$$

where the initial function satisfies $0 < u_0(x) \leq 1$ and $u_0 = 1$ on the boundary. Here T is a positive constant. We assume that the reaction term $f(u)$ is singular at $u = 0$ in the sense that $\lim_{u\downarrow 0} f(u) = -\infty$. For $u > 0$ we take $f(u)$ to be smooth and to satisfy $(-1)^k f^{(k)}(u) < 0$; $k = 0, 1, 2$.

It is well known, see, e.g., ([58] p.34, Th.3.3.), that the problem (1.4) has a local unique solution in a set $\Omega \times (0, t_\varepsilon)$. This solution can be continued to $\Omega \times (0,T)$, where $T = \inf_\tau\{\tau \geq 0 \quad | \limsup_{t\uparrow\tau, x\in\Omega}(u(x,t) + \frac{1}{u(x,t)}) = \infty\}$. It is also known that ([58] p.41, Th.3.8.) $u(x,t)$ is a C^∞-function with respect to x_i and t in $(x,t) \in \Omega \times (0,T)$.

We say that a is a quenching point and T is a quenching time for $u(x,t)$, if there exists a sequence $\{(x_n, t_n)\}$ with $x_n \to a$ and $t_n \uparrow T$, such that

$u(x_n, t_n) \to 0$ as $n \to \infty$. Correspondingly we say that b is a blowup point and T is a blowup time for $u(x,t)$, if there exists a sequence $\{(x_n, t_n)\}$ with $x_n \to b$ and $t_n \uparrow T$, such that $u(x_n, t_n) \to \infty$ as $n \to \infty$.

As a partial answer to the question (a) we have two kinds of sufficient conditions (with certain assumptions on f, u_0 and on the space dimension N):

Theorem 1.1. *When Ω is sufficiently large, then u quenches in finite time.*

Theorem 1.2. *When u_0 is small enough, then u quenches in finite time.*

In the first paper concerning the quenching problem, Kawarada [37] studied the equation (1.4), when $f(u) = -1/u$, $N = 1$ and $u_0 = 1$. Acker and Walter [2] obtained Theorem 1.1 for these type of singularities, when $u_0 = 1$. Essential in their work is the proof of the fact that for sufficiently large Ω, the problem (1.4) does not have a stationary solution.

Quenching can occur, even though the equation (1.4) does have a stationary solution. Then it is crucial that the initial function takes values which are close enough to zero. Acker and Kawohl [1] have proved Theorem 1.2, in the case where $N = 1$ and $\int_0^1 f(s)ds = \infty$. Levine [42] studies the stationary states of the problem (1.4), and proves Theorem 1.2, when $N = 1$ and $f(u) = -u^{-p}$ $(p > 0)$. In the proof the initial function u_0 is compared to the smallest stationary solution, and it is proved that this smallest stationary solution is unstable.

In the question (b) one studies the size of the set of quenching points. For example, it has been established for (1.4) that (under certain assumptions on u_0, $f(u)$ and N):

Theorem 1.3. *Quenching occurs at* $(0, T)$.

Theorem 1.4. *The set of quenching points is a compact subset of* Ω.

Theorem 1.5. *The set of quenching points is a discrete subset of* Ω.

Acker and Kawohl [1] proved Theorem 1.3 for functions $f(u)$ that satisfy $(-1)^k f^{(k)}(u) < 0$; $k = 0, 1, 2$; with Ω a ball in R^N, and with the initial function u_0 satisfying $\Delta u_0 + f(u_0) \leq 0$ and $(\Delta u_0 + f(u_0))_r \geq 0$. The argument is based on the inequality $u_{rt} \geq 0$, which is proved by the maximum principle.

Theorem 1.4 implies that quenching points are bounded away from the boundary. Deng and Levine [11] proved Theorem 1.4 in R^N under certain assumptions on $f(u)$ and u_0. They use the method developed in [21], where the corresponding blowup problem has been studied.

Guo [31] proved Theorem 1.5 for the case where $f(u) = -u^{-p}$, $(p > 0)$, and $u_0'' + f(u_0) \leq 0$ $(N = 1)$. The proof is based on Angenent's [3] result for certain parabolic equations.

As to the question (c), note that it is obvious that at least u_t or Δu in the equation blows up, when u quenches. Concerning the asymptotic behavior of solutions near a quenching point, the following results have been established (under various assumptions on u_0, $f(u)$ and N, note that in Theorem 1.6 $f(u)$ does not need to be a power singularity):

Theorem 1.6. *When u quenches, then u_t blows up.*

Theorem 1.7. *Let $f(u) = -u^{-p}$ $(p > 0)$. Then*

$$\min_{x\in\Omega} u(x,t) \leq [(1+p)(T-t)]^{1/(1+p)}$$

and

$$u(x,t) \geq C_1(T-t)^{1/(1+p)}$$

in a neighborhood of the quenching point $(t < T)$.

Theorem 1.8. *Let $f(u) = -u^{-p}$, $(p > 0)$. Then for any quenching point (a,T),*

$$\lim_{t\uparrow T} u(x,t)(T-t)^{-1/(1+p)} = (1+p)^{1/(1+p)}$$

uniformly, when $|x-a| \leq C\sqrt{T-t}$ for any positive constant C.

Let us present the corresponding results for the blowup problem (before we comment on Theorems 1.6-1.10). In this case, $f(u) = u^p$ or $f(u) = e^u$ in (1.4), with the boundary condition $u = 0$, when $x \in \partial\Omega$.

The results corresponding to Theorem 1.7 are

Theorem 1.9. *Let $f(u) = u^p$ and $N \geq 1$ in (1.4), then*

$$\max_{x\in\Omega} u(x,t) \geq \frac{c}{(T-t)^{1/(p-1)}}$$

and

$$u(x,t) \leq \frac{C}{(T-t)^{1/(p-1)}}$$

in a neighborhood of the blowup point $(t < T)$.

The results corresponding to Theorem 1.8 are (under certain assumptions on u_0 and N)

Theorem 1.10. *Let $f(u) = u^p$ in (1.4) and (a,T) be the blowup point. Then*

$$\lim_{t\uparrow T}(T-t)^{1/(p-1)}u(a+y\sqrt{T-t},t) = (p-1)^{-1/(p-1)},$$

uniformly, when $|y| \leq C$. When $f(u) = e^u$ in (1.4), then

$$\lim_{t\uparrow T}(u(a+y\sqrt{T-t},t) + \ln(T-t)) = 0,$$

uniformly for $|y| \leq C$.

Chan and Kwong [8] proved Theorem 1.6, for the case where $\int_0^1 f(u)du = \infty$. Deng and Levine [11] extended this Theorem to less singular reaction terms. Fila and Kawohl [15] proved Theorem 1.7. Note that then we obtain upper and lower bounds for $u(x,t)$, but that the upper bound is only valid at one point with respect to x.

Theorem 1.9 is from the paper by Friedman and McLeod [21].

The arguments behind Theorems 1.6, 1.7 and 1.9 are essentially based on the methods developed in [21]. By the maximum principle one can derive estimates in one direction (upper or lower bounds), and by the local existence theorem [58] one gets the opposite bounds at maximum or minimum points (with respect to the space variable).

Theorems 1.8 and 1.10 improve the results 1.7 and 1.9. More precisely, they give uniform estimates for $u(x,t)$ in backward parabolas of quenching and blowup points.

Note that the results concerning blowup problems were obtained earlier, and that the methods developed there have been applied to quenching problems.

Giga and Kohn [27, 28] proved Theorem 1.10 for $f(u) = u^p$. Their method is based on the scaling property of the equation $u_t - \Delta u = u^p$. This means that if $u(x,t)$ is a solution of the equation, then also the scaled functions

$$u_\lambda(x,t) = \lambda^{2\beta} u(\lambda x, \lambda^2 t), \tag{1.5}$$

with $\beta = 1/(p-1)$, $\lambda > 0$, are solutions of this equation. If $u_\lambda = u$ for all $\lambda > 0$, then u is said to be self-similar. If $(0,0)$ is a blowup point, then the asymptotics of $u(x,t)$ near the blowup point is given by u_λ, as $\lambda \to 0$. Here one defines new variables by $y = (-t)^{-\frac{1}{2}} x$ and $s = -\ln(-t)$, and then $w(y,s) = (-t)^\beta u(x,t)$. This function satisfies

$$w_s - w_{yy} + \frac{1}{2} y w_y + \beta w - w^p = 0. \tag{1.6}$$

For a solution w of(1.6), the self-similarity means that w does not depend on s. In the proof of Theorem 1.10 it is therefore essential to study the stationary solutions of (1.6). This analysis can be found in [27, 28]. In [28], Theorem 1.10 has been extended to a more general class of nonlinearities. More precisely, the results have been extended to reaction terms $f(u) = u^p + h(u)$, where $|h(u)| \leq b(1 + u^q)$ and $1 < q < p$.

Theorem 1.10 for $f(u) = e^u$ has been proved in [4] for the space dimension $N = 1, 2$ and in [5] for the space dimensions $N \geq 3$. These proofs are done by first defining new variables ($y = x/\sqrt{T-t}$, $s = -\ln(T-t)$), where now $w(y,s) = u(x,t) + \ln(T-t)$. Then it is crucial to show that $w \to w_0$, where w_0 is a solution of the stationary equation $w'' - \frac{1}{2} y w' + e^w - 1 = 0$, and finally to conclude the claim from the properties of this stationary equation.

Theorem 1.8 was first established by Guo [31], in the case where $N = 1$, $u_0''(x) - u_0^{-p}(x) \leq 0$ and $p \geq 3$. Fila and Hulshof [13] extended this result to $p \geq 1$. For the weaker singularities $0 < p < 1$, the proof is done in [33]. The extension of Theorem 1.10 to higher space dimensions has been worked out in [32] ($p \geq 1$) and in [14] ($p > 0$). Note also the paper [61] by Yuen, where a quenching rate-estimate for the degenerate equation,

$$x^q u_t - u_{xx} = -u^{-\beta},$$

with Cauchy-Dirichlet data in $\Omega = (0, a) \times (0, T)$, has been proved.

The proofs of Theorem 1.8 are based on methods developed by Giga and Kohn [27, 28]. The change of variables (assume that $(0, T)$ is the quenching point): $y = x/\sqrt{T-t}$, $s = -\ln(T-t)$ and $w(y,s) = (T-t)^{-1/(p+1)}u(x,t)$, now yields

$$w_s - w_{yy} + \frac{1}{2}yw_y + \frac{1}{p+1}w - w^{-p} = 0. \tag{1.7}$$

The study of the asymptotics when $t \uparrow T$ for the problem (1.4), when $f(u) = -u^{-p}$ is equivalent to having $s \to \infty$ in the equation (1.7). In the proof one shows that

(i) $w(y,s) \to w_\infty(y)$, as $s \to \infty$ (self-similarity),
and studies
(ii) the stationary equation

$$w'' - \frac{1}{2}yw' - F(w) = 0, \tag{1.8}$$

where $F(w) = \frac{1}{p+1}w - w^{-p}$. In (ii) one uses arguments based on [6], by which one can derive all possible limit functions w_∞ of w. The qualitative behavior of the solution of (1.8) depends essentially on the exponent p. By proving that a limit function w_∞ is constant, one obtains Theorem 1.8.

Another interesting question related to the asymptotics of solutions is whether one can refine the behavior of $u(x,t)$ in Theorem 1.8. More precisely, can one describe the shape of $u(x,t)$ with respect to the space variable in backward parabolas $|x-a| < C\sqrt{T-t}$, or can one determine how fast the limit value is reached? Moreover, one may ask whether the region $|x-a| < C\sqrt{T-t}$ can be enlarged? Because the domain of validity in Theorem 1.8 tends to zero as the quenching point is approached, then any information about the space structure of the solution at the quenching time T is lost. Therefore one needs more detailed information in order to be able to compute the profile of u at $t = T$. To this end, the following results have been established (under certain assumptions on u_0):

Theorem 1.11. *Let $f(u) = -u^{-p}$ $(p > 0)$ and let $(0, T)$ be the quenching point $(r = |x|, N \geq 1)$. Then*

$$u(r,T) \leq \Big[\frac{(p+1)^2}{2(1-p)}\Big]^{1/(1+p)} r^{2/(1+p)}, \qquad \textit{for } 0 < p < 1,$$

$$u(r,t) \geq C_\varepsilon r^{\varepsilon + 2/(1+p)}, \qquad \textit{for } 0 < p, \quad t \in (0,T].$$

Theorem 1.12. *Let $(0, T)$ be the quenching point for the equation (1.4), when $f(u) = -u^{-p}$ $(p > 0)$ and $N = 1$. Then for given $C > 0$ as $t \uparrow T$, either*

$$(T-t)^{-1/(p+1)}u(x,t) - (1+p)^{1/(1+p)} = \frac{(1+p)^{1/(1+p)}}{2p(-\ln(T-t))}\Big(x^2/2(T-t) - 1\Big) + o(1/(-\ln(T-t))),$$

or else, for some integer $m \geq 3$ and some constant $c \neq 0$

$$(T-t)^{-1/(p+1)}u(x,t) - (1+p)^{1/(1+p)} = \\ c(T-t)^{(m/2-1)}h_m(x/\sqrt{T-t}) + o((T-t)^{(m/2-1)}),$$

where the convergence takes place in $C^k(|x| < C\sqrt{T-t})$ for any $k \geq 0$. (h_m is the Hermite polynomial of order m)

Theorem 1.13. *Let $(0,T)$ be the quenching point for the equation (1.4), when $f(u) = -u^{-p}$ $(p > 0)$ and $N = 1$. Then*

$$u(x,T) = \Big[\frac{(p+1)^2}{8p}\Big]^{\frac{1}{1+p}}\Big(\frac{|x|^2}{|\ln|x||}\Big)^{\frac{1}{1+p}}(1+o(1)),$$

as $|x| \to 0$.

The result 1.11 is due to Fila and Kawohl [15]. They base their argument on an application of the maximum principle. See also the corresponding results for the blowup problem in [21]. Note that Theorem 1.11 tells us that for $p \in (0,1)$ the function $u(x,T)$ is of class C^1 at the origin, while for $p > 1$ it has a cusp-singularity and is merely Hölder continuous at the origin. However, this Theorem does not inform us about the exact profile of $u(x,t)$ at $t = T$.

Theorems 1.12 and 1.13 are from [18]. Theorem 1.12 was proved first, and was then used in the proof of Theorem 1.13. The method of the proof relies on corresponding blowup results, which were obtained in [19, 35, 59]. Note that Theorem 1.13 is also proved (independently) in [47], where, in addition, the stability of quenching problems is studied.

In the question (d) we are interested in the behavior of $u(x,t)$, when $t > T$. Because u_t blows up (Theorem 1.6), the equation (1.4) does certainly not have a strong solution for all $t > 0$. The answer to (d) therefore depends essentially on the concept of solution that one employs and also on how singular the reaction term is. It is interesting to know: (i) Whether the solution $u(x,t)$ can have nontrivial continuations when $t > T$? or (ii) Is $u(x,t)$ identically zero, when $t > T$ (complete quenching)? Note here that $f = f(u)\chi(\{u > 0\})$.

In [49] the singularity $-u^{-p}$ $(p \in (0,1))$ is regularized by the finite nonliearity $-u/(\varepsilon + u^{p+1})$. Then a classical global solution u_ε exists for every $\varepsilon > 0$. It is shown in [49] that u_ε is decreasing in ε, and that u_ε has a limit U as $\varepsilon \to 0$ which coincides with u for $t < T$. Moreover, it is proved that U is a global weak solution of $u_t - \Delta u = -u^{-p}\chi(\{u > 0\})$. Uniqueness of this U, however, is an open problem.

Properties of these weak solutions are further studied in [17], when $f = -\lambda u^{-p}\chi(\{u > 0\})$. In particular, it is established there that, for radially symmetric u_0, for λ sufficiently small and Ω a ball, then there is $t(u_0) \geq 0$ such that $U(x,t;u_0) > 0$ on $\overline{\Omega} \times (t(u_0), \infty)$.

In [24] the question (d) is analyzed for a larger class of singularities than in [17, 49]. A contribution in [24] consists in obtaining necessary and sufficient conditions for complete quenching depending on $f(u)$. For power singularities Galaktionov and Vazquez [24] have the following result.

Theorem 1.14. *Let* $f(u) = -u^{-p}$ *in* (1.4) *and* $N = 1$.

(a): Complete quenching occurs if and only if $p \geq 1$.

(b): If $0 < p < 1$, *then the solution of* (1.4) *has a non-trivial continuation after the quenching time* T.

The arguments in [24] are based on travelling-wave techniques. Solutions (in (b)) are viscosity solutions.

Finally we consider equations of type

$$v_t - v_{xx} = af(v) - bv_x^q \tag{1.9}$$

with Cauchy-Dirichlet data (v given on the boundary and $v(x, 0) = v_0(x)$), when $t > 0$ and $x \in \Omega$ (bounded). Here q, a and b are strictly positive constants, furthermore $f(v) = v^p$ or $f(v) = e^{rv}$ ($p, r > 0$ are constants). We have added a gradient damping term $-bv_x^q$ to the equation (1.4). The key question is, how this term affects the qualitative behavior of solutions. What conditions must q, a, b, $f(v)$ and Ω satisfy to guarantee that smooth solutions exist; alternatively, under what assumptions does blowup occur in finite time? These questions have been studied extensively (see for example [10, 40, 55, 56, 57] and references therein). Especially note [57], where the asymptotics of solutions have been investigated.

Consider now the equation (1.9), when $f(v) = e^{(1+p)v}$, $a = b = 1$ and $q = 2$. Substituting $v = -\ln(u)$ in the equation (1.4) ($N = 1$ and $f(u) = -u^{-p}$), we can see that $v(x, t)$ satisfies the equation (1.9). Therefore quenching for the equation (1.4) corresponds to blowup for the equation (1.9). This approach has been applied to study the blowup problem for the equation (1.9) in [1, 39, 40].

Detailed review-articles concerning the quenching problem are for example (Kawohl [38]) and (Levine [43]), and correspondingly on the blowup problem (Levine [44]) and (Galaktionov, Vazquez [25]). Furthermore, the blowup problem for the equation (1.9) has been studied in the review-article by Souplet, [56].

1.3 Motivation and results

Despite the existing rich literature, many open questions for the quenching and the blowup problem remain. Below we present the subjects that appear in this work.

(a) Does the singularity in the equation (1.4) necessarily lead to quenching, when the domain is large? Is it possible that there are weak singularities such that quenching cannot occur even for large domains?

(b) The second interesting question is to clear up whether quenching can take place on an entire interval, or is only possible on distinct points ($N = 1$)? Can the initial function u_0 be chosen such that the qualitative behavior differs from that of Theorem 1.5? On the other hand, can we have quenching on an entire interval for singularities different from power singularities?

(c) Quenching rate estimates (Theorem 1.8) have been proved only for power singularities and especially for equations that have the scaling property described in the preceding section. Does the mechanism of quenching (in the sense of Theorem 1.8) remain unchanged for every concave reaction term, i.e., $(-1)^k f^{(k)}(u) < 0$; $k = 0, 1, 2$?

(d) Refined asymptotic results, like Theorems 1.12 or 1.13, are known only for power singularities. If we are able to prove quenching rate estimates for certain other nonlinearities, can we refine these estimates? Or can we find some qualitative differences between the asymptotic behavior compared to the corresponding results in the case of the power singularities?

In the equation (1.4), an essential feature is the contest between the linear diffusion term Δu and the nonlinear reaction term $f(u)$. If the dissipative diffusion term is dominant, then there is no quenching. Thus the nonlinear reaction term can achieve quenching. Therefore the phenomenon is more interesting in the case of weaker singularities. Even if the corresponding stationary equation of (1.4) does not have a solution, then it might happen for a sufficiently weakly singular reaction term that quenching is only possible in infinite time.

A weakening of the nonlinearity in the equation (1.4) might lead to quenching on an entire interval. For power singularities we know by Theorem 1.5 that quenching occurs on distinct points which are bounded away from the boundary because of Theorem 1.4. Furthermore we have observed in the previous section that the solution of (1.4) loses less regularity at the quenching point in the case of weaker singularities. More precisely, by Theorem 1.13 it holds for $f(u) = -u^{-p}$ that the x-derivative of the final profile $u(x,T)$ at the quenching point has a singularity when $p \geq 1$ and is smooth ($u_x(a,T) = 0$), when $p \in (0,1)$. Can this regularity be strengthened for weaker nonlinearities in such a way that $u_x(x,T) = 0$ for all $x \in (c,b) \subset [-l,l]$, in other words can quenching take place on an entire interval?

The content of Theorem 1.8 can be interpreted by comparing the quenching rate to a solution of the corresponding ordinary differential equation $v' = f(v)$ (where $f(v) = -v^{-p}$, with final condition $v(T) = 0$), and concluding that these solutions are asymptotically equal in the region $|x-a| < C\sqrt{T-t}$.

We are now interested in whether this asymptotic equality holds for more general $f(u)$. More precisely, we conjecture that the quenching-rate satisfies

$$\lim_{t \uparrow T} \left(1 + \frac{1}{T-t} \int_0^{u(x,t)} \frac{d\tau}{f(\tau)} \right) = 0 \tag{1.10}$$

uniformly, when $|x-a| < C\sqrt{T-t}$ for every $C \in (0,\infty)$. By Theorem 1.8 this holds for power singularities. Can the equality (1.10) be obtained for the solution of the equation (1.4) in the situation where the scaling property is not valid. In particular we are interested in the validity of this formula for weak singularities. We stated earlier that crucial to the occurrence of quenching is the competition between the linear diffusion term and the nonlinear reaction term, the nonlinear term promoting quenching. Inasmuch as we are now interested in less singular $f(u)$, it is not at all obvious that (1.10) remains true.

Before we introduce the results of this work, we wish to give one further motivation for our approach which consists in the study of quenching problems with weak singularities. This motivation is a possible extension of Theorem 1.14. By this Theorem it follows that weak singularities do allow nontrivial continuations for $t > T$, whereas strong singularities do not. Obviously, this fact makes weak singularities particularly interesting and fruitful to study.

1.4 Paper I

We begin our analysis by considering the problem

$$\begin{aligned} u_t - u_{xx} = \ln(\alpha u), \qquad x \in (-l, l), \quad t \in (0, T), \\ u(x, 0) = u_0(x), \quad x \in [-l, l], \\ u(\pm l, t) = 1, \quad t \in [0, T), \end{aligned} \tag{1.11}$$

where $\alpha \in (0, 1)$ and $u_0 \in (0, 1]$. The reaction term $f(u) = -u^{-p}$ in the equation (1.4) is here replaced by the weaker logarithmic singularity $f(u) = \ln(\alpha u)$. The space dimension is one. The quenching problem for the equation (1.11) is the subject of our first paper [50]. Below we give a brief overview of the results.

Because the reaction term is now much weaker than a power singularity, it is not obvious that quenching can happen. The first problem is therefore to clear up, whether quenching may at all occur?

It is assumed throughout [50] that the initial function satisfies

$$u_0''(x) + \ln(\alpha u_0(x)) \leq 0, \tag{1.12}$$

where $x \in [-l, l]$. This technical assumption guarantees that $u(x, t)$ is decreasing in time. It is shown in [50] that quenching is possible, i.e., we have:

Theorem 1.15. *For l large enough, the solution $u(x, t)$ of* (1.11) *quenches in finite time.*

The proof of this theorem is based on the fact that the stationary problem corresponding to the equation (1.11) has no solution, if l is sufficiently large.

The second result in [50] concerns the potential quenching on an entire interval. We study whether the weakening of the singularity affects the size of the set of quenching points. The result in [50] tells us that the situation does not qualitatively differ from the situation, where we had a power singularity.

Theorem 1.16. *Suppose that $u(x, t)$ satisfies* (1.11) *and that* (1.12) *holds. Then the set of quenching points is finite.*

The proof is based on a general method for certain parabolic equations developed by Angenent [3]. It is first deduced, using this method, that u_x cannot oscillate, when the quenching point is approached. Then it is shown that there is a time t^* such that there is a finite number of local minima with respect to x after t^*, and that this number is constant in time. Finally, one shows that quenching cannot occur on the boundary and that the set of quenching points is finite. The proof is essentially the same as Guo's [31] (which is also based on [3]) for the stronger singularities $f(u) = -u^{-p}$.

The main result in [50] treats the local asymptotics of the solution near the quenching point. This is formulated as

Theorem 1.17. *Let $u(x,t)$ be the solution of the equation* (1.11)*, where u_0 is even, $u_0'(r) \geq 0$, $u_0(x) \in (0,1]$ and* (1.12) *holds ($r = |x|$). Assume that $u(x,t)$ quenches at $(0,T)$ for some $T < \infty$. Then*

$$\lim_{t \uparrow T}(1 + \frac{1}{T-t}\int_0^{u(x,t)} \frac{d\tau}{\ln(\alpha\tau)}) = 0, \tag{1.13}$$

uniformly, when $|x| < C\sqrt{T-t}$, for every $C \in (0,\infty)$.

This theorem can also be proved in a somewhat stronger form:

Corollary 1.18. *Let $u(x,t)$ quench at (a,T), with an initial function u_0 that satisfies $u_0(x) \in (0,1]$ and* (1.12)*. Then*

$$\lim_{t \uparrow T}(1 + \frac{1}{T-t}\int_0^{u(x,t)} \frac{d\tau}{\ln(\alpha\tau)}) = 0,$$

uniformly, when $|x-a| < C\sqrt{T-t}$, for every $C \in (0,\infty)$.

Our proof of the quenching rate estimate (1.13) for a logarithmic singularity is not based on earlier results on quenching. The proof here uses similarity variables and energy estimates; in particular observe that our method is different from the earlier versions used to prove the corresponding quenching-rate estimate (1.10) (see Giga-Kohn [27, 28], Bebernes-Eberly [5], Guo [31]). This is already a consequence of the fact that (1.11) does not have the useful scaling property that the equation (1.4) (with $f(u) = -u^{-p}$) has.

By these quenching results we can also study the blowup for the gradient damping equation of type (1.9). More precisely, substituting $\alpha u = e^{-v}$ in the equation (1.11), we get

$$\begin{gathered} v_t - v_{xx} = \alpha v e^v - v_x^2, \qquad x \in (-l,l), \quad t \in (0,T), \\ v(x,0) = -\ln(\alpha u_0(x)), \quad x \in [-l,l], \\ v(\pm l, t) = -\ln(\alpha), \quad t \in [0,T), \end{gathered} \tag{1.14}$$

Note that quenching for the equation (1.11) corresponds to blow-up in the equation (1.14). Thus Theorems 1.15, 1.16 and 1.17 yield the following new Corollaries.

Corollary 1.19. *For sufficiently large l, the solution $v(x,t)$ of* (1.14) *blows up in finite time.*

Corollary 1.20. *The set of blow-up points for the equation* (1.14) *is finite.*

Corollary 1.21. *Let $(0,T)$ be a blow-up point for the equation* (1.14)*. Then*

$$\lim_{t\uparrow T} \frac{1}{T-t} \int_{v(x,t)}^{\infty} \frac{d\tau}{\alpha \tau e^{\tau}} = 1,$$

uniformly, when $|x| \le C\sqrt{T-t}$.

1.5 Paper II

In the paper [51] we extend Theorem 1.17 to a wider class of weak singularities. More precisely, we assume that $f(u)$ in (1.4) (where $N=1$) satisfies

$$|u^n f^{(n)}(u)| = o(|f(u)|), \quad n = 1, 2, \tag{1.15}$$

as $u \downarrow 0$. Furthermore we define $\tilde{f}(s) = -e^s \cdot \frac{f(e^{-s})}{f'(e^{-s})}$, and assume that

$$\tilde{f}(s(1+o(1))) = (1+o(1))\tilde{f}(s), \tag{1.16}$$

as $s \to \infty$. Explicitly, this requirement means that for $a(s) \to 0$ as $s \to \infty$, there is $b(s) \to 0$ as $s \to \infty$, such that $\tilde{f}(s(1+a(s))) = (1+b(s))\tilde{f}(s)$ as $s \to \infty$. Note that (1.15) implies $\tilde{f}(s) \to \infty$ as $s \to \infty$. The hypothesis (1.16) refines the asymptotic character of (1.15). More precisely, it imposes a condition on the possible oscillations of $f(u)/(uf'(u))$ in a neighborhood of $u=0$.

The main result of this paper is

Theorem 1.22. *Let $N=1$ and $u_0'' + f(u_0) \le 0$ in* (1.4)*. Assume that* (1.15) *and* (1.16) *hold and that u quenches. Then the quenching rate satisfies the estimate* (1.10)*.*

An interesting ingredient in our result is the fact that the class of nonlinearities is now additive with a mild restriction. More precisely, we can verify that if two singularities $f_1(u)$ and $f_2(u)$ satisfy (1.15), then also $f(u) = -|f_1(u)|^p - |f_2(u)|^q$, for $p, q > 0$ satisfies (1.15). The condition (1.16) on slow variation does not necessarily obey this rule. However, we can give several sufficient conditions for $f_1(u)$ and $f_2(u)$, such that $f(u) = f_1(u) + f_2(u)$ satisfies (1.16), provided that $f_1(u)$ and $f_2(u)$ do. For example, we can take f_1 and f_2 such that $|f_2| = o(|f_1|)$ and $|f_2'| = o(|f_1'|)$, to guarantee that (1.16) is additive. In this sense our result is new compared with earlier works on uniform quenching-rate estimates.

Let us give some examples that satisfy (1.15) and (1.16). Because (1.15) and (1.16) hold for $f(u) = \ln(u)$, then by the above remarks these hypotheses hold also for $f(u) = -|\ln(u)|^p, \quad p > 0$ or $f(u) = -|\ln(u)|^p - |\ln(u)|^q$, $(p, q >$

0). Furthermore we can derive that $f(u) = -\ln(|\ln(u)|)$ or even $f(u) = -\ln(|\ln|\ln|\cdots(|\ln(u)|)\cdots|||)$ satisfy (1.15) and (1.16). Consequently $f(u) = -|\ln(u)|^p - \ln(|\ln(u)|)$, $(p > 0)$, and $f(u) = -\ln(|\ln(u)|) - \ln(\ln(|\ln(u)|))$ are as well suitable.

Stronger singularities, like $f(u) = -u^{-p}, \quad p > 0$ or $f(u) = -\frac{u^{-p}}{|\ln(u)|}, \quad p > 0$, do not satisfy (1.15).

Finally we emphasize that nonlinearities can be perturbed in many ways. We can take $h \in C^2[0,1]$ such that $h > 0$, and then find that $f(u) = h(u)\ln(u)$ satisfies (1.15) and (1.16).

In this paper we first show that quenching occurs for sufficiently large l in (1.4), where (1.15) and (1.16) holds. Then we prove (1.10) for these nonlinearities.

1.6 Paper III

In the paper [52] we refine the asymptotic result (1.13). The main result, Theorem 1.23 below, gives a precise asymptotic expression for the solution in a backward space-time parabola near a quenching point. The analysis is based on methods developed in [18, 19, 35, 59]. These techniques were first developed for blowup problems of reaction diffusion equations in [19, 35]. Subsequently these approaches were applied to quenching problems with a power singularity in [18].

We briefly explain how Theorem 1.23 is proved. We first conclude by Theorem 1.17 that

$$\lim_{t\uparrow T}\Big(\frac{u(x,t)}{(T-t)(-\ln(T-t))} - 1\Big) = 0, \tag{1.17}$$

uniformly when $|x| < C\sqrt{T-t}$. Then we define y and s as earlier, and let $\phi(y,s)$ be the left-hand side of (1.17). Substituting this in the equation (1.11), we deduce that $\phi(y,s)$ satisfies

$$\phi_s = \mathcal{L}\phi + \frac{1}{s}f(\phi) + g(s), \tag{1.18}$$

where $f(\phi) = \ln(1+\phi) - \phi$, $g(s) = \frac{\ln(s)}{s}(1+o(1))$ and $\mathcal{L} = \frac{\partial^2}{\partial y^2} - \frac{y}{2}\frac{\partial}{\partial y} + 1$.

We study the equation (1.18) as a dynamical system in the space $L^2_\rho(R)$, where $\rho(y) = \exp(-y^2/4)$. Therefore we expand the function $\phi(y,s)$ with respect to the eigenfunctions of $\mathcal{L}$ in that space, i.e., $\phi = \sum a_j(s)h_j(y)$. Here the functions $h_j(y)$ are the scaled Hermite polynomials which form an orthonormal base on $L^2_\rho(R)$. The spectrum of this operator is $\lambda_j = \frac{2-j}{2}$, where $j = 0,1,2,..$ By projecting the equation (1.18) to the subspaces generated by the functions $h_j(y)$, we get the ordinary differential equations for $a_j(s)$:

$$a_j'(s) = (1-\frac{j}{2})a_j(s) + \langle\frac{f(\phi)}{s} + g, h_j\rangle_{L^2_\rho} \quad j = 0,1,2,... \tag{1.19}$$

By analogy with classical ODE theory, we expect that one term in the Fourier series is dominant, i.e., $\phi(y,s) \approx a_j(s)h_j(y)$, for some j, as $s \to \infty$.

Linearizing for the nonzero eigenvalues, we get $\phi(y,s) \approx c_j \exp\left(\frac{2-j}{2}s\right)h_j(y)$. The positive eigenvalues ($j = 0, 1$) are incompatible with the result (1.17), and therefore the nonlinear part has to dominate the positive eigenspace in (1.19). For the zero eigenvalue ($j = 2$), we can see that the linear part vanishes. Moreover, after some calculations we obtain that $a_2(s)$ satisfies:

$$a_2'(s) = -c^* \frac{1}{s}(1 + o(1))a_2(s)^2,$$

from which we obtain after integration that $\phi(y,s) \approx \frac{C^*}{\ln(s)}\big(y^2 - 2\big)$.

In [52] we give a proof for this formal argument. The presence of a nontrivial null space for the operator $\mathcal{L}$ suggests the use of center manifold theory. More precisely, we use the methods developed in [18, 19, 35, 59] for the analysis of infinite dimensional dynamical systems.

The main result of this paper gives a refined asymptotics of the quenching:

Theorem 1.23. *Assume that* (1.12) *holds for the equation* (1.11) *and that* $u(x,t)$ *quenches at* $(0,T)$. *Assume further that* $|a_2(s)| \geq M(\ln(s)/s)^2$ *for some* $M > 0$. *Then for any* $C > 0$ *and* $\varepsilon > 0$ *there exists* t_0 *such that*

$$\sup_{|x|<C\sqrt{T-t}} \Big| \frac{u(x,t)}{(T-t)(-\ln(T-t))} - 1 - \frac{1}{8\ln(-\ln(T-t))}\big(\frac{x^2}{T-t} - 2\big) \Big| = O\big(\frac{\varepsilon}{\ln(-\ln(T-t))}\big), \tag{1.20}$$

when $t \in [t_0, T)$.

1.7 Paper IV

In the paper [53] we study the quenching problem for the equation (1.4), when $x \in \Omega = B_R(0) = \{y \in \mathcal{R}^N; |y| < R\}$. We take the initial function u_0 to be radial. Then the solution is also radial and $\Delta = \frac{\partial^2}{\partial r^2} + \frac{(N-1)\partial}{r\partial r}$. Our goal is to extend the results in [50, 51, 52] with some addition to the N-dimensional situation.

The first goal is to show that quenching occurs for reaction terms of type (1.15) and (1.16) also in this N-dimensional setting. Because the term $\frac{(N-1)\partial u}{r\partial r}$ resists quenching in the equation (1.4), it is not obvious that quenching actually takes place if $N > 1$. However, we show that for sufficiently large domains Ω quenching occurs in finite time. The proof is very similar to that of the case $N = 1$ [51].

The second ingredient is the quenching rate-estimate (1.10). For the same reason as above, this is not at all a direct consequence of the one dimensional result (1.10) (in [51]) which was proved for the singularities of type (1.15) and (1.16) in [51]. In this paper we derive this result for the N-dimensional situation using basically the same method as in [50, 51]. However there are differences at the technical level in the proof which are given in [53].

The third objective of this paper is to study the refined asymptotics of the solution near the quenching point. In [52], the analysis concerned the equation (1.11). We now extend the approach to a class of nonlinearities that satisfy (1.15) and (1.16). More precisely, we study the refined asymptotic behavior of the quenching for the equation (1.4), where

(i) $f(u) = -|\ln(u)|^p$, $(p > 0)$

(ii) $f(u) = -|\ln(u)|^p - |\ln(u)|^q$, $p \geq q + 1$, $p > 1$ and $q > 0$.

Under certain assumptions we give a proof of the result corresponding to Theorem 1.23.

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