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GEHRING LEMMA IN METRIC SPACES

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Abstract: *We present a proof for the Gehring lemma in a metric measure space endowed with a doubling measure. As an application we show the self improving property of Muckenhoupt weights as well as higher integrability of Jacobians of quasisymmetric mappings.*

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1 Introduction

The following self improving property of the reverse Hölder inequality is a result of Gehring [3]. Assume that a non–negative locally integrable function satisfies the inequality

$$\left(\fint_B f^p dx\right)^{1/p} \le c \fint_B f dx \tag{1.1}$$

for all balls B of $\mathbb{R}^n$, for a constant c and $1 < p < \infty$. Then there exists $\varepsilon > 0$ such that

$$\left(\fint_B f^{p+\varepsilon} dx\right)^{1/p+\varepsilon} \le c \fint_B f dx \tag{1.2}$$

for some other constant c. It is generally known that the theorem remains true also in a metric space equipped with a doubling measure. However, the proof is slightly difficult to find in the literature.

The subject has been studied for example by Fiorenza [2] as well as D'Apuzzo and Sbordone [1], [10]. In Gianazza [4] it is shown that if a function satisfies (1.1), then there exists $\varepsilon > 0$ such that

$$\left(\fint_X f^{p+\varepsilon} d\mu\right)^{1/p+\varepsilon} \le c \fint_X f d\mu \tag{1.3}$$

for some constant c. The result is obtained in a space of homogeneous type, provided that $0 < \mu(X) < \infty$. Also Kinnunen examines various minimal and maximal inequalities and reverse Hölder inequalities in [8].

Likewise in a doubling metric measure space, Strömberg and Torchinsky prove Gehring's result under the additional assumption that the measure of a ball depends continuously on its radius, see [11]. Zatorska–Goldstein [12] proves a version of the lemma, where on the right–hand side there is a ball with a bigger radius.

We present a proof of the Gehring lemma in a doubling metric measure space. Our method is classical and intends to be as transparent as possible. In particular, we obtain the result for balls in the sense of (1.2) in the metric setting instead of (1.3). The proof is based on a Calderón–Zygmund type argument which produces a bigger ball on the right–hand side of (1.2). However, the measure induced by a function satisfying the reverse Hölder inequality turns out to be doubling.

As an application we study Jacobians of quasisymmetric mappings and Muckenhoupt weights on metric spaces. As a corollary we prove higher integrability of the volume derivative, where we follow the presentation of Heinonen and Koskela [7]. Finally, we show that the Muckenhoupt class is an open ended condition. The proof is classical.

2 General Assumptions

Let (X, d, μ) be a metric measure space equipped with a Borel regular measure μ such that the measure of every nonempty open set is positive and that the measure of every bounded set is finite.

Our notation is standard. We assume that a ball B in X comes always with a fixed centre and radius, i.e. $B = B(x,r) = \{y \in X : d(x,y) < r\}$ with $0 < r < \infty$. We denote

$$u_B = \fint_B u\,d\mu = \frac{1}{\mu(B)} \int_B u\,d\mu,$$

and when there is no possibility for confusion we denote kB the ball $B(x,kr)$. We assume in addition that μ is *doubling* i.e. there exists a constant c_d such that

$$\mu(B(x,2r)) \le c_d \mu(B(x,r))$$

for all balls B in X. We refer to this property by calling (X,d,μ) a doubling metric measure space. This is different from the concept of *doubling space.* The latter is a property of the metric space (X,d), where all balls can be covered by a constant number of balls with radius half of the radius of the original ball. A doubling metric measure space is always doubling as a metric space.

A good reference for the basic properties of a doubling metric measure space is [6]. In particular, we will need two elementary facts. Concider a ball containing disjoint balls such that their radii are bounded below. In a doubling space the number of these balls is bounded. Secondly, μ being doubling implies that for all pairs of radii $0 < r \le R$ the inequality

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \le c_d \left(\frac{R}{r}\right)^Q$$

holds true for all $x \in X$. Here $Q = \log_2 c_d$ is sometimes called the *doubling dimension* of (X,d,μ).

Throughout the paper, constants are generally denoted c and they may not be the same everywhere. However, if not otherwise mentioned, they depend only on fixed constants such as those associated with the structure of the space, the doubling constant etc.

3 Gehring lemma

Throughout this section we suppose that (X,d,μ) is doubling and we denote it briefly X.

Theorem 3.1 (Gehring lemma). *Let $1 < p < \infty$ and $f \in L^1_{loc}(X)$ be non–negative. If there exists a constant c such that f satisfies the reverse Hölder inequality*

$$\left(\fint_B f^p d\mu\right)^{1/p} \le c \fint_B f\,d\mu \tag{3.1}$$

for all balls B of X, then there exists $q > p$ such that

$$\left(\fint_B f^q d\mu\right)^{1/q} \le c_q \fint_B f\,d\mu \tag{3.2}$$

for all balls B of X. The constant c_q as well as q depend only on the doubling constant, p, and on the constant in (3.1).

Let us first prove that a function satisfying the reverse Hölder inequality defines a doubling measure. This property turns out to be essential in the proof of Theorem 3.1.

Proposition 3.2. *Let $f \in L^1_{loc}(X)$ be a non–negative function that satisfies the reverse Hölder inequality* (3.1)*. Then the measure induced by f is doubling, i.e.*

$$\int_{2B} f d\mu \le c \int_B f d\mu$$

for all balls B of X. The constant c depends only on the constant in (3.1).

Proof. Define

$$\nu(U) = \int_U f d\mu$$

for $U \subset X$ μ–measurable. Fix a ball $B \subset X$ and let $E \subset B$ be a μ–measurable set. Then

$$\begin{aligned}\int_B f\chi_E d\mu &\le \left(\int_B f^p d\mu\right)^{1/p} \mu(E)^{1-1/p} \\ &\le \left(\int_B f d\mu\right) \mu(B)^{1/p-1}\mu(E)^{1-1/p} = c\nu(B)\left(\frac{\mu(E)}{\mu(B)}\right)^{1-1/p}.\end{aligned}$$

The inequalities above follow from the Hölder and the reverse Hölder inequalities, respectively. This implies

$$\frac{\nu(E)}{\nu(B)} \le c\left(\frac{\mu(E)}{\mu(B)}\right)^{1/p'} \tag{3.3}$$

for all $E \subset B$ and p' the L^p–conjugate exponent of p. Since the set E in (3.3) is arbitrary, we can replace it by $B \setminus E$. Therefore

$$\frac{\nu(B \setminus E)}{\nu(B)} \le c\left(\frac{\mu(B \setminus E)}{\mu(B)}\right)^{1/p'},$$

which is equivalent to

$$1 - \frac{\nu(E)}{\nu(B)} \le c\left(1 - \frac{\mu(E)}{\mu(B)}\right)^{1/p'} \tag{3.4}$$

for all $E \subset B$. If $E = \alpha B$, then by choosing $0 < \alpha < 1$ small enough

$$c\left(1 - \frac{\mu(\alpha B)}{\mu(B)}\right)^{1-1/p'} < \frac{1}{2} \tag{3.5}$$

holds true. It follows from (3.4) and (3.5) that

$$1 - \frac{\nu(\alpha B)}{\nu(B)} < \frac{1}{2}$$

and hence $\nu(B) \geq 2\nu(\alpha B)$. We are now able to iterate this. There exists $k \in \mathbb{N}$ such that $\alpha^k < 1/2$ and thus

$$\nu(B) \leq 2v(\alpha B) \leq 2^k \mu(\alpha^k B) \leq 2^k \nu(\frac{1}{2}B)$$

for all balls B of X. This proves that ν is doubling. Remark that μ being doubling plays no role here. $\square$

The following is a standard iteration lemma, see [5].

Lemma 3.3. *Let $Z : [R_1, R_2] \subset \mathbb{R} \to [0, \infty)$ be a bounded non–negative function. Suppose that for all ρ, r such that $R_1 \leq \rho < r \leq R_2$*

$$Z(\rho) \leq \big(A(r-\rho)^{-\alpha} + B(r-\rho)^{-\beta} + C\big) + \theta Z(r) \tag{3.6}$$

holds true for some constants $A, B, C \geq 0$, $\alpha > \beta > 0$ and $0 \leq \theta < 1$. Then

$$Z(R_1) \leq c(\alpha, \theta)\big(A(R_2 - R_1)^{-\alpha} + B(R_2 - R_1)^{-\beta} + C\big). \tag{3.7}$$

Lemma 3.3 is needed in the proof of our first key lemma:

Lemma 3.4. *Let $R > 0$, $q > 1$, $k > 1$ and $f \in L^q_{loc}(X)$. If for all $0 < r \leq R$ and for an arbitrary constant c*

$$\fint_{B(x,r)} f^q d\mu \leq \varepsilon \fint_{B(x,kr)} f^q d\mu + c \left(\fint_{B(x,kr)} f d\mu\right)^q \tag{3.8}$$

holds, then

$$\fint_{B(x,R)} f^q d\mu \leq c \left(\fint_{B(x,2R)} f d\mu\right)^q, \tag{3.9}$$

if $\varepsilon > 0$ is small enough. The constant in (3.9) depends on the doubling constant and on the constant in (3.8).

Proof. Fix $R > 0$ and choose $r, \rho > 0$ such that $R \leq \rho < r \leq 2R$. Set $\tilde{r} = (r - \rho)/k$. Now

$$B(x, \rho) \subset \bigcup_{y \in B(x,\rho)} B(y, \tilde{r}/5)$$

and by the Vitali covering theorem there exist disjoint balls $\{B(x_i, \tilde{r}/5)\}_{i=1}^{\infty}$ such that $x_i \in B(x, \rho)$ and

$$B(x, \rho) \subset \bigcup_{i=1}^{\infty} B(x_i, \tilde{r}).$$

These balls can be chosen in a way that

$$\sum_{i=1}^{\infty} \chi_{B(x_i, k\tilde{r})} \leq M \tag{3.10}$$

for some constant $M < \infty$. This follows from the doubling property of the space. Indeed, assume that y belongs to N balls $B(x_i, k\tilde{r})$. Clearly

$$B(x_i, k\tilde{r}) \subset B(y, 2k\tilde{r}) \subset B(y, 2R).$$

Remember that $\tilde{r}$ and R are fixed and choose $K = 20R/\tilde{r}$. Now there are N disjoint balls with radius $B(x_i, \tilde{r}/5) \geq 2R/K$ included in a fixed ball $B(y, 2R)$. Since the space is doubling, we must have $N \leq M(K)$. The inequality (3.10) follows.

Observe then that by the doubling property and the construction of the balls $\{B(x_i, \tilde{r})\}_i$ we have

$$\begin{aligned}\sum_i \mu(B(x_i, \tilde{r})) &\leq c \sum_i \mu(B(x_i, \tilde{r}/5)) = c\mu(\cup_i B(x_i, \tilde{r}/5)) \\ &\leq c\mu(B(x, r)) \leq c \left(\frac{r}{\rho}\right)^Q \mu(B(x, \rho)).\end{aligned}$$

On the other hand $B(x, \rho) \subset B(x_i, 2k\rho)$, so that

$$\begin{aligned}\mu(B(x, \rho)) \leq \mu(B(x_i, 2k\rho)) &\leq c \left(\frac{2k\rho}{\tilde{r}}\right)^Q \mu(B(x_i, \tilde{r})) \\ &= c \left(\frac{\rho}{r - \rho}\right)^Q \mu(B(x_i, \tilde{r})).\end{aligned}$$

Combining these two inequalities implies

$$\begin{aligned}\mu(B(x, \rho)) &\geq c \left(\frac{r}{\rho}\right)^{-Q} \sum_i \mu(B(x_i, \tilde{r})) \\ &\geq \left(\frac{r}{\rho}\right)^{-Q} \left(\frac{\rho}{r - \rho}\right)^{-Q} \sum_i \mu(B(x_i, \rho)).\end{aligned}$$

And as a consequence

$$\#\{B(x_i, \tilde{r})\} \leq c \left(\frac{r}{\rho}\right)^Q \left(\frac{\rho}{r - \rho}\right)^Q,$$

i.e. the number of balls $B(x_i, \tilde{r})$ is at most $c\big(r/(r - \rho)\big)^Q$, where c depends only on the doubling constant and $Q = \log_2 c_d$.

Observe that (3.8) holds true for $\tilde{r}$, so that

$$\begin{aligned}\int_{B(x_i, \tilde{r})} f^q d\mu &\leq \varepsilon \frac{\mu(B(x_i, \tilde{r}))}{\mu(B(x_i, k\tilde{r}))} \int_{B(x_i, k\tilde{r})} f^q d\mu \\ &\quad + c \frac{\mu(B(x_i, \tilde{r}))}{\mu(B(x_i, k\tilde{r}))^q} \left(\int_{B(x_i, k\tilde{r})} f d\mu\right)^q \\ &\leq \varepsilon \int_{B(x_i, k\tilde{r})} f^q d\mu + c\mu(B(x_i, \tilde{r}))^{1-q} \left(\int_{B(x_i, k\tilde{r})} f d\mu\right)^q \qquad (3.11)\end{aligned}$$

because μ is doubling. We note that

$$\frac{\mu(B(x,r))}{\mu(B(x_i,\tilde{r}))} \le \frac{\mu(B(x_i,2r))}{\mu(B(x_i,r))} \le c_d \left(\frac{2r}{\tilde{r}}\right)^Q \le c\left(\frac{r}{r-\rho}\right)^Q,$$

from which it follows that

$$\mu(B(x_i,\tilde{r}))^{1-q} \le c\left(\frac{r}{r-\rho}\right)^{Q(q-1)} \mu(B(x,r))^{1-q}.$$

Together with (3.11) this implies

$$\begin{aligned}\int_{B(x_i,\tilde{r})} f^q d\mu &\le \varepsilon \int_{B(x_i,k\tilde{r})} f^q d\mu \\ &\quad + c\left(\frac{r}{r-\rho}\right)^{Q(q-1)} \mu(B(x,r))^{1-q} \left(\int_{B(x_i,k\tilde{r})} f d\mu\right)^q. \quad (3.12)\end{aligned}$$

Since $B(x,\rho) \subset \cup_i B(x_i,\tilde{r})$, summing over i in (3.12) gives

$$\begin{aligned}\int_{B(x,\rho)} f^q d\mu &\le \sum_i \int_{B(x_i,\tilde{r})} f^q d\mu \\ &\le \varepsilon \sum_i \int_{B(x_i,k\tilde{r})} f^q d\mu \\ &\quad + c\left(\frac{r}{r-\rho}\right)^{Q(q-1)} \mu(B(x,r))^{1-q} \sum_i \left(\int_{B(x_i,k\tilde{r})} f d\mu\right)^q \\ &\le \varepsilon M \int_{B(x,r)} f^q d\mu \\ &\quad + c\left(\frac{r}{r-\rho}\right)^{Q(q-1)} \mu(B(x,r))^{1-q} \left(\frac{r}{r-\rho}\right)^Q \left(\int_{B(x,r)} f d\mu\right)^q \\ &= \varepsilon M \int_{B(x,r)} f^q d\mu + c\left(\frac{r}{r-\rho}\right)^{Qq} \mu(B(x,r))^{1-q} \left(\int_{B(x,r)} f d\mu\right)^q.\end{aligned}$$

Finally, remember that $R \le \rho < r \le 2R$, so that

$$\begin{aligned}\int_{B(x,\rho)} f^q d\mu &\le \varepsilon M \int_{B(x,r)} f^q d\mu \\ &\quad + cR^{Qq}(r-\rho)^{-Qq} \mu(B(x,r))^{1-q} \left(\int_{B(x,r)} f d\mu\right)^q\end{aligned}$$

and furthermore

$$\begin{aligned}\fint_{B(x,\rho)} f^q d\mu &\le \varepsilon c \int_{B(x,r)} f^q d\mu \\ &\quad + cR^{Qq}(r-\rho)^{-Qq} \mu(B(x,r))^{1-q} \left(\fint_{B(x,2R)} f d\mu\right)^q. \quad (3.13)\end{aligned}$$

We are able to iterate this. In Lemma 3.3 set

$$Z(\rho) := \fint_{B(x,\rho)} f^q d\mu,$$

so that Z is bounded on $[R, 2R]$. Set also $R_1 = R$, $R_2 = 2R$, $\alpha = Qq$ and

$$A = cR^{Qq}\left(\fint_{B(x,2R)} f d\mu\right)^q > 0,$$

where c is the constant in (3.13). Putting $\theta = c\varepsilon$ and choosing ε so small that $c\varepsilon < 1$, (3.13) satisfies the assumptions of Lemma 3.3 with $B = C = 0$. This yields $Z(R) \le cA(2R - R)^{-Qq}$, that is

$$\begin{aligned}\fint_{B(x,R)} f^q &\le cR^{Qq}(cR - R)^{-Qq}\left(\fint_{B(x,2R)} f d\mu\right)^q \\ &= c\left(\fint_{B(x,2R)} f d\mu\right)^q.\end{aligned}$$

□

In the following we consider the Hardy–Littlewood maximal function restricted to a fixed ball $100B_0$, that is

$$Mf(x) = \sup_{\substack{B \ni x \\ B \subset 100B_0}} \fint_B f d\mu.$$

Clearly the coefficient 100 can be replaced by any other sufficiently big constant. The role of this constant is setting a playground large enough to assure that all balls we are dealing with stay inside this fixed ball. The basic tools of analysis we use work for this maximal function as well.

Lemma 3.5. *Let f be a non–negative function in $L^1_{loc}(X)$ and satisfy the reverse Hölder inequality* (3.1)*. Then for all balls B in X*

$$\int_{\{x \in B \,:\, Mf(x) > \lambda\}} f^p d\mu \le c\lambda^p \mu(\{x \in 100B \,:\, Mf(x) > \lambda\}), \tag{3.14}$$

for all $\lambda > \operatorname{ess\,inf}_B Mf$ with some constant depending only on p, the doubling constant and on the constant in 3.1.

Proof. Let us fix a ball B_0 with radius $r_0 > 0$. We denote $\{x \in X \,:\, Mf(x) > \lambda\}$ briefly by $\{Mf > \lambda\}$. Let $\lambda > \operatorname{ess\,inf}_B Mf$. Now there exists $x \in B_0$ so that $Mf(x) \le \lambda$. This implies that $B_0 \cap \{Mf \le \lambda\} \ne \emptyset$. For every $x \in B_0 \cap \{Mf > \lambda\}$ set

$$r_x = \operatorname{dist}(x, 100B_0 \setminus \{Mf > \lambda\}),$$

so that $B(x, r_x) \subset 100B_0$. Remark that the radii r_x are uniformly bounded by $2R$.

In the consequence of the Vitali covering theorem there are disjoint balls $\{B(x_i, r_{x_i})\}_{i=1}^{\infty}$ such that

$$B_0 \cap \{Mf > \lambda\} \subset \bigcup_{i=1}^{\infty} 5B_i,$$

where we denote $B_i = B(x_i, r_i)$. Both $B_i \subset 100B_0$ and $5B_i \subset 100B_0$ for all $i = 1, 2, \ldots$, so they are still balls of (X, d). Furthermore, $5B_i \cap \{Mf \leq \lambda\} \neq \emptyset$ for all $i = 1, 2, \ldots$ so that

$$\fint_{5B_i} f d\mu \leq Mf(x) \leq \lambda \tag{3.15}$$

for all $i = 1, 2, \ldots$. We can now estimate the integral on the left side in (3.14). A standard estimation shows that

$$\begin{aligned}\int_{B_0 \cap \{Mf > \lambda\}} f^p d\mu &\leq \int_{\cup_i 5B_i} f^p d\mu \leq \sum_i \int_{5B_i} f^p d\mu \\ &= \sum_i \mu(5B_i) \fint_{5B_i} f^p d\mu \leq c^p \sum_i \mu(5B_i) \left(\fint_{5B_i} f d\mu \right)^p \\ &\leq c^p \lambda^p \sum_i \mu(5B_i),\end{aligned}$$

where the last inequality follows from the reverse Hölder inequality and the second last from (3.15). Since μ is doubling and the balls B_i disjoints we get

$$\sum_i \mu(5B_i) \leq c \sum_i \mu(B_i) = c\mu(\cup_i B_i).$$

By definition $B_i \subset 100B_0 \cap \{Mf > \lambda\}$ for all $i = 1, 2, \ldots$. Therefore

$$\int_{B_0 \cap \{Mf > \lambda\}} f^p d\mu \leq c\lambda^p \mu(\cup_i B_i) \leq c\lambda^p \mu(100B_0 \cap \{Mf > \lambda\}$$

for all $\lambda > \operatorname{ess\,inf}_{B_0} Mf$. $\square$

Remark. Note that $\operatorname{ess\,inf}_{B_0} Mf \neq \infty$.

In the well known weak type estimate for locally integrable functions

$$\mu(B_0 \cap \{Mf > \lambda\}) \leq \frac{c}{\lambda} \int_{100B_0} f d\mu,$$

the right–hand side tends to zero when $\lambda \to \infty$. The constant c depends only on the doubling constant c_d. We can thus choose $0 < \lambda_0 < \infty$ so that

$$\frac{c}{\lambda_0} \int_{100B_0} f d\mu \leq \frac{1}{2} \mu(B_0).$$

As a consequence

$$\begin{aligned}\mu(B_0 \cap \{Mf \leq \lambda_0\}) &= \mu(B_0) - \mu(B_0 \cap \{Mf > \lambda_0\}) \\ &\geq \mu(B_0) - \frac{c}{\lambda_0} \int_{100B_0} f d\mu \geq \frac{1}{2}\mu(B_0).\end{aligned}$$

This leads to $\operatorname{ess\,inf}_{B_0} Mf \leq \lambda_0$, for if $\operatorname{ess\,inf}_{B_0} Mf > \lambda_0$, then $Mf(x) > \lambda_0$ for almost every $x \in B_0$. This impossible since $\mu(B_0 \cap \{Mf \leq \lambda_0\}) \geq \frac{1}{2}\mu(B_0)$.

For the reader's convenience we present here one technical part of our proof as a separate lemma.

Lemma 3.6. *Let $1 < q < \infty$ and $f \in L^q_{loc}(X)$. Suppose in addition that f satisfies the reverse Hölder inequality. Then for every $1 < p < q$*

$$\int_{B \cap \{Mf > \alpha\}} f^q d\mu \leq c\alpha^q \mu(100B \cap \{Mf > \alpha\}) + c\frac{q-p}{q} \int_{100B} (Mf)^q d\mu, \quad (3.16)$$

where $\alpha = \operatorname{ess\,inf}_B Mf$ and c depends on p, the doubling constant and on the constant in 3.1.

Proof. Fix a ball $B_0 \subset X$. Let $\alpha = \operatorname{ess\,inf}_{B_0} Mf$, so that $Mf \geq \alpha$ μ–a.e. on $100B_0$. Set $d\nu = f^p d\mu$. Now

$$\int_{B_0 \cap \{Mf > \alpha\}} f^q d\mu = \int_{B_0 \cap \{Mf > \alpha\}} f^{q-p} f^p d\mu \leq \int_{\{Mf > \alpha\}} (Mf)^{q-p} d\nu.$$

However, for every positive measure and measurable function g and set E

$$\int_E g^p d\nu = p \int_0^\infty \lambda^{p-1} \nu\left(\{x \in E \,:\, |g(x)| > \lambda\}\right) d\lambda$$

for all $0 < p < \infty$. This implies

$$\begin{aligned}\int_{B_0 \cap \{Mf > \alpha\}} f^q d\mu &\leq (q-p) \int_0^\infty \lambda^{q-p-1} \nu(B_0 \cap \{Mf > \alpha\} \cap \{Mf > \lambda\}) d\lambda \\ &= (q-p) \int_0^\alpha \lambda^{q-p-1} \nu(B_0 \cap \{Mf > \alpha\}) d\lambda \\ &\quad + (q-p) \int_\alpha^\infty \lambda^{q-p-1} \nu(B_0 \cap \{Mf > \lambda\}) d\lambda.\end{aligned}$$

Replacing $d\nu = f^p d\mu$ and integrating the first integral over λ we get

$$\begin{aligned}\int_{B_0 \cap \{Mf > \alpha\}} f^q d\mu \leq &\int_{B_0 \cap \{Mf > \alpha\}} \alpha^{q-p} f^p d\mu \\ &+ (q-p) \int_\alpha^\infty \lambda^{q-p-1} \int_{B_0 \cap \{Mf > \lambda\}} f^p d\mu d\lambda.\end{aligned}$$

We can now use Lemma 3.5 on both integrals on the right–hand side and get

$$\int_{B_0\cap\{Mf>\alpha\}} f^q d\mu \le c\alpha^q \mu(100B_0 \cap \{Mf > \alpha\}) \\ + c(q-p)\int_\alpha^\infty \lambda^{q-1}\mu(100B_0 \cap \{Mf > \lambda\})d\lambda.$$

Then by changing the order of integration we arrive at

$$\begin{aligned}\int_{B_0\cap\{Mf>\alpha\}} f^q d\mu &\le c\alpha^q \mu(100B_0 \cap \{Mf > \alpha\}) \\ &\quad + c(q-p)\int_\alpha^\infty \lambda^{q-1}\int_{100B_0\cap\{Mf>\lambda\}} d\mu d\lambda \\ &= c\alpha^q \mu(100B_0 \cap \{Mf > \alpha\}) \\ &\quad + c(q-p)\int_{100B_0}\int_\alpha^{Mf} \lambda^{q-1} d\lambda d\mu,\end{aligned}$$

from which by integrating the second term over α we conclude that

$$\begin{aligned}\int_{B_0\cap\{Mf>\alpha\}} f^q d\mu &\le c\alpha^q \mu(100B_0 \cap \{Mf > \alpha\}) \\ &\quad + c\frac{q-p}{q}\int_{100B_0} \big((Mf)^q - \alpha\big) d\mu \\ &\le c\alpha^q \mu(100B_0 \cap \{Mf > \alpha\}) \\ &\quad + c\frac{q-p}{q}\int_{100B_0} (Mf)^q d\mu.\end{aligned}$$

□

Finally, before starting the proof of our main theorem we recall the following property of maximal functions.

Lemma 3.7. *Let $f \in L^p_{loc}(X)$, $1 < p < \infty$. Then there is a constant c depending only on p and c_d, such that*

$$\int_B (Mf)^p d\mu \le c\int_B f^p d\mu$$

for all balls B of X.

Proof of the Gehring lemma. Consider a fixed ball B_0. Set $\alpha = \operatorname{ess\,inf}_{B_0} Mf$ and let $q > p$ be an arbitrary real number for the moment. We divide the integral of f^q over B_0 into two parts:

$$\int_{B_0} f^q d\mu = \int_{B_0\cap\{Mf>\alpha\}} f^q d\mu + \int_{B_0\cap\{Mf\le\alpha\}} f^q d\mu. \tag{3.17}$$

The second integral in (3.17) is easier to estimate, and we have

$$\int_{B_0\cap\{Mf\le\alpha\}} f^q d\mu \le \int_{B_0\cap\{Mf\le\alpha\}} (Mf)^q d\mu \le \alpha^q \mu(100B_0 \cap \{Mf \le \alpha\}).$$

It would be tempting to use Lemma 3.6 to the second integral in (3.17), but this would require $f \in L^q_{loc}(X)$. Unfortunately that is exactly what we need to prove. The function f is assumed to be locally integrable and by the reverse Hölder inequality it is also in the local L^p–space. Nevertheless, we can replace f with the truncated function $f_i = \min\{f, i\}$. The reverse Hölder inequality (3.1), Lemmas 3.5, 3.6 and Proposition 3.7 as well as the preceeding analysis hold for f_i. In addition, $f_i \in L^q_{loc}(X)$. We continue to denote the function f but remember that from now on we mean the truncated function.

With (3.16) we get now from (3.17)

$$\begin{aligned}\int_{B_0} f^q d\mu &\le c\alpha^q \mu(100B_0) \cap \{Mf > \alpha\}) + c\frac{q-p}{q}\int_{100B_0} (Mf)^q d\mu \\ &\quad + \alpha^q \mu(100B_0) \cap \{Mf \le \alpha\}) \\ &\le c\alpha^q \mu(100B_0) + c\frac{q-p}{q}\int_{100B_0} (Mf)^q d\mu\end{aligned}$$

and furthermore

$$\fint_{B_0} f^q d\mu \le c\alpha^q + c\frac{q-p}{q}\fint_{100B_0} (Mf)^q d\mu.$$

This is true for all $q > p$. Let $\varepsilon > 0$ and choose $q > p$ such that $c(q-p)/p < \varepsilon$. Then

$$\fint_{B_0} f^q d\mu \le c\alpha^q + \varepsilon\fint_{100B_0} (Mf)^q d\mu. \tag{3.18}$$

Now that $f = f_i$ is locally q–integrable, the equation (3.18) gives

$$\fint_{B_0} f^q d\mu \le c\alpha^q + \varepsilon\fint_{100B_0} f^q d\mu \tag{3.19}$$

due to Proposition 3.7. We had chosen α such that $\alpha \le Mf$ for μ–a.e. x in B_0. Hence

$$\begin{aligned}\alpha^p = \fint_{B_0} \alpha^p d\mu &\le \fint_{B_0} (Mf)^p d\mu \le c\fint_{100B_0} (Mf)^p d\mu \\ &\le c\fint_{100B_0} f^p d\mu \le c\left(\fint_{100B_0} f d\mu\right)^p,\end{aligned}$$

where we use again Proposition 3.7 and the reverse Hölder inequality. Moreover

$$\alpha^q \le c\left(\fint_{100B_0} f d\mu\right)^q. \tag{3.20}$$

From (3.19) and (3.20) we conclude that

$$\fint_{B_0} f^q d\mu \le \varepsilon \fint_{100B_0} f^q d\mu + c\left(\fint_{100B_0} f d\mu\right)^q \tag{3.21}$$

for all balls B_0 of X. If necessary, choose a smaller ε and thus also a q closer to p in (3.18) to make Lemma 3.4 hold true. Set $k = 100$ in the lemma to obtain

$$\fint_{B_0} f^q d\mu \le c\left(\fint_{2B_0} f d\mu\right)^q.$$

Since f satisfies the reverse Hölder inequality and the measure $\int f d\mu$ is doubling, we have

$$\begin{aligned}\fint_{B_0} f^q d\mu &\le c\left(\frac{1}{\mu(2B_0)}\int_{2B_0} f d\mu\right)^q \le c\left(\frac{1}{\mu(2B_0)}\int_{B_0} f d\mu\right)^q \\ &\le c\left(\fint_{B_0} f d\mu\right)^q.\end{aligned}$$

It remains to pass to the limit with $i \to \infty$ and the theorem follows. $\square$

4 Volume derivative of quasisymmetric mappings – higher integrability

In this section we study quasisymmetric mappings between two metric spaces X and Y. We show that the volume derivative of a quasisymmetric mapping, i.e.

$$\mu_f(x) = \lim_{r\to 0} \frac{|f(B(x,r))|}{|B(x,r)|},$$

is higher integrable. In the Euclidean setting μ_f equals to the Jacobian of f. We follow closely the presentation of Heinonen and Koskela [7]. For the basic properties of quasisymmetric mappings on metric spaces we also refer to [6].

For this section we introduce a notation $|\cdot - \cdot| = d(\cdot,\cdot)$ for the metric on Y.

4.1 Definitions

We begin by recalling some definitions and properties of quasisymmetric mappings.

Definition 4.1. *Let $u : X \to Y$ be a function. A non–negative Borel measurable function $g : X \to [0,\infty]$ is said to be an upper gradient of u if for all rectifiable paths γ joining points x and y we have*

$$|u(x) - u(y)| \le \int_\gamma g ds.$$

Definition 4.2. *Let $1 \leq p < \infty$. We say that (X, d, μ) admits a weak $(1,p)$–Poincaré inequality if there exist constants $\tau \geq 1$ and $c_p \geq 1$ such that*

$$\fint_B |u - u_B| d\mu \leq c_p (\operatorname{diam} B) \left(\fint_{\tau B} g^p d\mu \right)^{1/p}$$

for all balls B of X, for all functions $u : X \to [0, \infty]$ integrable in τB and for all upper gradients of u.

Definition 4.3. *A space (X, d, μ) is Q–regular if there is a constant $c \geq 1$ such that*

$$\frac{1}{c} r^Q \leq \mu(B(x,r)) \leq c r^Q$$

for all $x \in X$ and $0 < r < \operatorname{diam} X$.

Definition 4.4. *A homeomorphism between two metric spaces X and Y is said to be η–quasisymmetric if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that*

$$|x - a| \leq t|x - b| \ \Rightarrow \ |f(x) - f(a)| \leq \eta(t) |f(x) - f(b)|$$

for all $t > 0$ and $a, b, x \in X$.

It turns out that η has to be increasing and $\eta(0) = 0$.

Proposition 4.5. *If $f : X \to Y$ is quasisymmetric and if $A_1 \subset A_2 \subset X$ are such that $0 < \operatorname{diam} A_1 \leq \operatorname{diam} A_2 < \infty$, then $\operatorname{diam} f(A_2)$ is finite and*

$$\frac{1}{2\eta\left(\frac{\operatorname{diam} A_2}{\operatorname{diam} A_1}\right)} \leq \frac{\operatorname{diam} f(A_1)}{\operatorname{diam} f(A_2)} \leq \eta\left(\frac{2 \operatorname{diam} A_1}{\operatorname{diam} A_2}\right).$$

For the proof of Proposition 4.5 we refer to [6].

Proposition 4.6. *Let $f : X \to Y$ be quasisymmetric. Then for all $x \in X$ and $r > 0$ there exist two constants $0 < r_x < R_x$ such that*

$$B(f(x), r_x) \subset f(B(x,r)) \subset B(f(x), R_x). \tag{4.1}$$

Proof. Fix $B = B(x,r)$ in X with $r > 0$. Since $f(B)$ is bounded by quasisymmetry it is sufficient to show the existence of $r_x > 0$ such that $B(f(x), r_x) \subset f(B)$.

Let $r_x > 0$ be arbitrary for the moment. The function $f : X \to Y$ is η–quasisymmetric, so that $f^{-1} : f(X) \to X$ is η'–quasisymmetric, where $\eta'(t) = 1/\eta^{-1}(t^{-1})$ for $t > 0$. This implies that

$$B(f(x), r_x) \subset f(B) \ \Leftrightarrow \ f^{-1}(B(f(x), r_x)) \subset B.$$

We set $A_1 = B(f(x), r_x)$ and $A_2 = B$ in Lemma 4.5 and obtain

$$\frac{1}{2\eta'\left(\frac{\operatorname{diam} f(B)}{2r_x}\right)} \leq \frac{\operatorname{diam} f^{-1}(B(f(x), r_x))}{2r} \leq \eta'\left(\frac{4r_x}{\operatorname{diam} f(B)}\right). \tag{4.2}$$

A sufficient but not a necessary condition for $f^{-1}(B(f(x), r_x))$ to be contained in $B(x, r))$ is that

$$\operatorname{diam} f^{-1}(B(f(x), r_x)) < r. \tag{4.3}$$

From (4.2) we can deduce

$$\operatorname{diam} f^{-1}(B(f(x), r_x)) \leq 2\eta' \left(\frac{4r_x}{\operatorname{diam} f(B)} \right) r.$$

Therefore, if we choose $r_x > 0$ such that

$$\eta' \left(\frac{4r_x}{\operatorname{diam} f(B)} \right) < \frac{1}{4},$$

the assertion follows. $\square$

4.2 Higher Integrability of the Volume Derivative

From now on, let X and Y be Q–regular metric measure spaces with $Q > 1$ that are doubling and rectifiably connected i.e. all points can be joined by a rectifiable curve. We denote the Hausdorff Q–measure in both spaces by $\mathcal{H}_Q$ and write

$$|A| = \mathcal{H}_Q(A), \qquad dx = d\mathcal{H}_Q(x).$$

Proposition 4.7. *In the above setting the measure*

$$\nu(E) = |f(E)|$$

is doubling on X, when f is quasisymmetric and $E \subset X$ measurable.

Proof. Note first that the Hausdorff measure in a Q–regular space is doubling. Indeed, for all $x \in X$ and $r > 0$ we have

$$|B(x, 2r)| \leq c(2r)^Q \leq c|B(x, r)|$$

by Q–regularity. Then, let us fix a ball B_0. By Proposition 4.6 there exist $0 < r_0 < R_0$ for the ball B_0 and $0 < r_2 < R_2$ for the ball $2B_0$ such that (4.1) holds and especially $r_0 < R_2$ because $f(B_0)$ is included in $f(2B_0)$. Then

$$|f(2B_0)| \leq |B(f(x), R_2)| \leq c|B(f(x), r_0)| \leq c|f(B_0)|,$$

where we use also the doubling property of the Hausdorff measure. $\square$

Definition 4.8. *Suppose that $f : X \to Y$ is a quasisymmetric homeomorphism. Define the volume derivative in $x \in X$ as*

$$\mu_f(x) = \lim_{r \to 0} \frac{|f(B(x, r))|}{|B(x, r)|}.$$

The spaces X and Y are Q–quasiregular, so Q is also their Hausdorff dimension. Therefore the measures $|f(\cdot)|$ and $|\cdot|$ are finite for all compact subset of X and Y and thus Radon measures. In addition, both measure spaces are doubling, so that the volume derivative of f exists, is finite for a.e. $x \in X$ and and is locally integrable satisfying

$$\int_E \mu_f(x)dx \leq |f(E)| \tag{4.4}$$

for every measurable set E of X. For further discussion in the Euclidean setting, see [9]. The analysis remains true in a doubling metric measure space.

Definition 4.9. *Suppose that $f : X \to Y$ is a quasisymmetric map. Define*

$$L_f(x,r) = \sup_{y \in B(x,r)} |f(x) - f(y)| \tag{4.5}$$

and the maximum derivative of f as

$$L_f(x) = \limsup_{r \to 0} \frac{L_f(x,r)}{r}, \tag{4.6}$$

that describes the local stretching of f. L_f is a Borel regular function in X.

Proposition 4.10. *For all $x \in X$ and $r > 0$ the inequalities*

$$L_f(x,r)^Q \leq |f(B(x,r))| \tag{4.7}$$

and

$$c' L_f(x)^Q \leq \mu_f(x) \leq c L_f(x)^Q \tag{4.8}$$

hold with constants depending only on the doubling constant, the constant associated with Q–regularity and on η in Definition 4.4.

Proof. Let B be an arbitrary ball with radius $r > 0$ and center $x \in X$. Proposition 4.6 implies that there exist $0 < r_x < R_x$ such that

$$B(f(x), r_x) \subset f(B(x,r)) \subset B(f(x), R_x).$$

The Hausdorff measure is doubling in X (see the proof of Proposition 4.7), and hence

$$|B(f(x), r_x)| \geq c_d^{-1} \left(\frac{r_x}{R_x}\right)^Q |B(f(x), R_x)|.$$

Set $c_0 = c_d^{-1}(r_x/R_x)^Q$ and note that now c_0 depends only on η and on the doubling constant. It follows that

$$|f(B(x,r))| \geq |B(f(x), r_x)| \geq c_0 |B(f(x), R_x)| \geq c R_x^Q$$

by Q–regularity. In addition it is clear that

$$(2R_x)^Q \geq (\operatorname{diam} f(B(x,r)))^Q \geq |f(x) - f(y)|^Q$$

for all $y \in B(x,r)$, and therefore

$$|f(B(x,r))| \geq c \sup_{y \in B(x,r)} |f(x) - f(y)|^Q.$$

The inequality (4.7) follows.

Q–regularity and (4.7) imply now that

$$\left(\frac{L_f(x,r)}{r}\right)^Q \leq c\frac{|f(B(x,r)|}{|B(x,r)|},$$

from which the first inequality in (4.8) follows by letting r tend to zero. The second inequality does not require f being quasisymmetric, only the Q–regularity of X. Indeed, note first that $\operatorname{diam} f(B(x,r)) \leq cL_f(x,r)$. Then by Q–regularity

$$\begin{aligned}\mu_f(x) = \limsup_{r \to 0} \frac{|f(B(x,r)|}{|B(x,r)|} &\leq c \limsup_{r \to 0} \frac{|f(B(x,r)|}{r^Q} \\ &\leq c \limsup_{r \to 0} \left(\frac{L_f(x,r)}{r}\right)^Q.\end{aligned}$$

The equation (4.8) follows. □

In the following, let $f : X \to Y$ be an η-quasisymmetric map. For $\varepsilon > 0$ define

$$L_f^\varepsilon(x) = \sup_{0 < r \leq \varepsilon} \frac{L_f(x,r)}{r}.$$

Now L_f^ε decreases as ε decreases, and

$$\lim_{\varepsilon \to 0} L_f^\varepsilon(x) = L_f(x)$$

for all $x \in X$.

Lemma 4.11. *There is a constant c such that for each $\varepsilon > 0$, the function cL_f^ε is an upper gradient of the function $u(x) = |f(x) - f(x_0)|$.*

Proof. Fix a ball B with a radius $r < \operatorname{diam} X$. Fix $\varepsilon > 0$, and let γ be a rectifiable curve joining two points x and y in B. Set $d = \operatorname{diam} \gamma$. If $z \in \gamma$ is arbitrary, then

$$\operatorname{diam}(f(\gamma)) \leq 2L_f(z,d). \tag{4.9}$$

The proof of (4.9) follows directly from definitions. Indeed,

$$\begin{aligned}\operatorname{diam}(f(\gamma)) &= \sup_{y_1,y_2 \in \gamma} |f(y_1) - f(y_2)| \\ &\leq \sup_{y_1,y_2 \in \gamma} \big(|f(y_1) - f(z)| + |f(z) - f(y_2)|\big) \\ &\leq \sup_{y_1,y_2 \in B(z,d)} \big(|f(y_1) - f(z)| + |f(z) - f(y_2)|\big) \\ &= 2L_f(z,d).\end{aligned}$$

Suppose first that $d \le \varepsilon$. Then

$$L_f^\varepsilon(z) = \sup_{0<r\le\varepsilon} \frac{L_f(z,r)}{r} \ge \frac{L_f(z,d)}{d} \ge \frac{\operatorname{diam}(f(\gamma))}{2d}.$$

Therefore

$$\int_\gamma L_f^\varepsilon ds \ge \int_\gamma \frac{L_f(z,d)}{d} ds \ge \ell(\gamma)\frac{\operatorname{diam}(f(\gamma))}{2d},$$

where $\ell(\gamma)$ is the length of γ. Clearly $|f(x)-f(y)| \le \operatorname{diam} f(\gamma)$ and $d \le \ell(\gamma)$, and hence

$$\begin{aligned}\int_\gamma L_f^\varepsilon ds &\ge \frac{\ell(\gamma)\big|f(x)-f(y)\big|}{2d} \ge \frac{1}{2}\Big|\big|f(x)-f(x_0)\big| - \big|f(x_0)-f(y)\big|\Big| \\ &= \frac{1}{2}\big|u(x)-u(y)\big|,\end{aligned}$$

i.e. $2L_f^\varepsilon$ is an upper gradient of u.

Suppose then that $d > \varepsilon$. Since γ is rectifiable, $l(\gamma) < \infty$ and we can pick successive points $x_0, \ldots, x_N$ from γ such that $x = x_0 < x_1 < \ldots < x_N = y$ and such that for each $i = 1, \ldots, N$ $\operatorname{diam}(\gamma_i) < \varepsilon$, where γ_i is the portion of γ between x_{i-1} and x_i. Now we can proceed as in the first case:

$$\begin{aligned}\int_\gamma L_f^\varepsilon ds &= \sum_{i=1}^N \int_{\gamma_i} L_f^\varepsilon ds \ge \sum_{i=1}^N \frac{1}{2}\big|f(x_i)-f(x_{i-1})\big| \\ &\ge \frac{1}{2}\big|f(x)-f(y)\big| \ge \frac{1}{2}\big|u(x)-u(y)\big|.\end{aligned}$$

This finishes the proof. $\square$

Lemma 4.12. *Let B be an arbitrary ball with a radius r_0 in X. The function L_f^ε belongs to space weak–$L^Q(B)$ with norm independent of ε provided that ε is small enough. More precisely, for $\varepsilon < r_0/10$ and $t > 0$ we have that*

$$|\{x \in B \,:\, L_f^\varepsilon(x) > t\}| \le ct^{-Q}|f(B)|,$$

where $c \ge 1$ depends only on η and the data of X and Y. A fortiori, the function L_f belongs to weak–$L^Q(B)$ with a norm depending only on the data.

Proof. We begin by noting that the set

$$E_t = \{x \in B \,:\, L_f^\varepsilon(x) > t\}$$

is open, so that

$$E_t \subset \bigcup_{x\in E_t} B(x, r_x).$$

By the Vitali covering theorem we can then find a countable collection of disjoint balls $\{B(x_i, r_i)\}_{i=1}^\infty$ such that $0 < r_i \le \varepsilon$,

$$\frac{L_f(x_i, r_i)}{r_i} > t \tag{4.10}$$

and that

$$E_t \subset \bigcup_i 5B_i \subset 2B$$

provided that ε is small enough. We denote $B_i = B(x_i, r_i)$. Recall the definition of the Hausdorff measure

$$|E_t| = \lim_{\delta \to 0} \inf_{\mathcal{B}} \sum_{B \in \mathcal{B}} \operatorname{diam}(B)^Q,$$

where the infimum is taken over all covers $\mathcal{B}$ of E_t by balls of diameter at most δ. Hence

$$|E_t| \leq c \sum_i r_i^Q \leq ct^{-Q} \sum_i L_f(x_i, r_i)^Q \leq ct^{-Q} \sum_i |f(B_i)|$$

by (4.7) and (4.10). The balls B_i are disjoint, so it follows that

$$|E_t| \leq ct^{-Q}|f(\cup_i B_i)| \leq ct^{-Q}|f(2B)| \leq ct^{-Q}|f(B)|.$$

The last inequality follows from the fact that the measure defined by f is doubling since f is quasisymmetric. Note that the sets $f(B_i)$ and $f(B_j)$ are disjoint for all pairs $i \neq j$ for the reason that B_i and B_j are disjoint and f is a homeomorphism. Since $L_f \leq L_f^\varepsilon$, the claim follows. □

Theorem 4.13. *Suppose that X and Y are locally compact Q–regular spaces for some $Q > 1$ and that X admits a weak $(1,p)$–Poincaré inequality for some $p < Q$. Let f be a quasisymmetric map from X to Y. Then there exist a constant c and $\varepsilon > 0$ such that*

$$\left(\fint_B \mu_f^{1+\varepsilon} dx \right)^{1/(1+\varepsilon)} \leq c \fint_B \mu_f dx \tag{4.11}$$

for all balls $B \subset X$. The constant c depends only on the quasisymmetry function η of f, on the constants associated with the Q–regularity of X and Y and on the constant in the Poincaré inequality.

Proof. Let us fix a ball $B = B(x_0, r)$. The function $u(x) = |f(x) - f(x_0)|$ is bounded and continuous in B since f is a homeomorphism. Therefore u is integrable in B. Set $B' = \tau^{-1}B$, $r' = \tau^{-1}r$ and note that by Lemma 4.11 L_f^ε is an upper gradient of u. Then by the Poincaré inequality

$$\fint_{B'} |u - u_{B'}| dx \leq cr \left(\fint_B (L_f^\varepsilon)^p dx \right)^{1/p}$$

and letting ε tend to zero we get

$$\fint_{B'} |u - u_{B'}| dx \leq cr \left(\fint_B (L_f)^p dx \right)^{1/p}. \tag{4.12}$$

On the other hand

$$\begin{aligned}
u_{B'} &= \fint_{B'} |f(x) - f(x_0)|dx \\
&= \frac{1}{|B'|}\int_{B'\setminus\frac{1}{2}B'} |f(x) - f(x_0)|dx + \frac{1}{|B'|}\int_{\frac{1}{2}B'} |f(x) - f(x_0)|dx \\
&\geq \frac{1}{|B|}\int_{B'\setminus\frac{1}{2}B'} |f(x) - f(x_0)|dx.
\end{aligned}$$

In the last inequality we are able to make the estimation

$$L(x_0, r) \leq c|f(x) - f(x_0)|$$

in $B' \setminus \frac{1}{2}B'$. Indeed, $|x_0 - x| \geq r'/2$, and moreover $|x - y| \leq 2r' \leq c_0|x_0 - x|$ for all $x \in B' \setminus \frac{1}{2}B'$ and $y \in B$. By the definition of quasisymmetry this implies

$$|f(x) - f(y)| \leq \eta(c_0)|f(x) - f(x_0)|$$

for all $x \in B' \setminus \frac{1}{2}B'$ and $y \in B$. Here we are forced to pay more attention to constants.

$$\begin{aligned}
L(x_0, r) &= \sup_{y\in B} |f(x_0) - f(y)| \leq \sup_{y\in B} \left(|f(x_0) - f(x)| + |f(x) - f(y)| \right) \\
&\leq \sup_{y\in B} \Big(|f(x_0) - f(x)| + \eta(c_0)|f(x_0) - f(x)| \Big) \leq c_1|f(x_0) - f(x)|,
\end{aligned}$$

where $c_1 = \max\{1, \eta(c_0)\}$. Using this in (4.13) we get

$$u_{B'} \geq c_1^{-1}\frac{|B' \setminus \frac{1}{2}B'|}{|B'|}L(x_0, r). \tag{4.13}$$

Next we claim that

$$|f(x) - f(x_0)| \leq \eta(\delta)L_f(x_0, r) \tag{4.14}$$

for all $x \in \delta B'$ if $0 < \delta < r$. To see this, let $x \in \delta B'$ and $y \in B \setminus B'$ (if $B = B'$, take for example $y \in B \setminus \frac{r-\delta}{2}B$). Then $|x - x_0| < \delta r' \leq \delta|y - x_0|$, and quasisymmetry implies

$$|f(x) - f(x_0)| \leq \eta(\delta)|f(y) - f(x_0)|$$

for all $y \in B \setminus \frac{1}{2}B'$. Consequently

$$|f(x) - f(x_0)| \leq \eta(\delta) \sup_{y\in B\setminus\frac{1}{2}B'} |f(y) - f(x_0)| \leq \eta(\delta)L_f(x_0, r).$$

Remember that η is increasing and $\eta(0) = 0$, so it is possible to choose $\delta > 0$ such that $\eta(\delta) \leq (2c_1)^{-1}$. This leads to

$$u(x) \leq \eta(\delta)L_f(x_0, r) \leq (2c_1)^{-1}L_f(x_0, r) \tag{4.15}$$

for all $x \in \delta B'$. Combining (4.13) and (4.15) we get

$$|u(x) - u_{B'}| \geq |c_1^{-1} L_f(x_0, r) - (2c_1)^{-1} L_f(x_0, r)| = (2c_1)^{-1} L_f(x_0, r)$$

for all $x \in \delta B'$, and therefore

$$\begin{aligned}\int_{B'} |u(x) - u_{B'}| dx &\geq \int_{\delta B'} |u(x) - u_{B'}| dx \geq (2c_1)^{-1} L_f(x_0, r) |\delta B'| \\ &\geq c L_f(x_0, r) |B'|.\end{aligned} \tag{4.16}$$

Now the Poincaré inequality (4.12) together with (4.16) implies

$$\frac{L_f(x_0, r)}{r} \leq \frac{c}{r} \fint_{B'} |u(x) - u_{B'}| dx \leq c \left(\fint_B L_f^p dx \right)^{1/p}.$$

This estimate enables us to prove the reverse Hölder inequality for L_f with exponents (Q, p). Indeed,

$$\left(\fint_B L_f^Q dx \right)^{1/Q} \leq c \left(\fint_B \mu_f dx \right)^{1/Q} \leq c \left(\frac{|f(B)|}{|B|} \right)^{1/Q}$$

by (4.4) and (4.8). The inequality $|f(B)| \leq (\operatorname{diam} f(B))^Q$ follows directly from the definition of the Hausdorff measure. Furthermore,

$$\operatorname{diam} f(B) = \sup_{x, y \in B} |f(x) - f(y)| \leq 2 L_f(x_0, r),$$

so that

$$\left(\frac{|f(B)|}{|B|} \right)^{1/Q} \leq \left(\frac{2 L_f(x_0, r)}{\frac{1}{c} r^Q} \right)^{1/Q} \leq c \left(\fint_B L_f^p dx \right)^{1/p}$$

by Proposition 4.10 and (4.17). This proves that

$$\left(\fint_B L_f^Q dx \right)^{1/Q} \leq c \left(\fint_B L_f^p dx \right)^{1/p} \tag{4.17}$$

for all balls B in X.

Next we use (4.8) to prove the reverse Hölder ineequality for μ_f. Denote $g = L_f^p$, so that $g^{Q/p} = L_f^q$. The equation (4.17) can be written in this notation as

$$\left(\fint_B g^t dx \right)^{1/t} \leq c \fint_B g dx,$$

where $t = Q/p > 1$. By the Gehring lemma 3.1 there exists $\delta > 0$ such that

$$\left(\fint_B g^{t+\delta} dx \right)^{1/(t+\delta)} \leq c \fint_B g dx \tag{4.18}$$

for all balls B in X. In Proposition 4.10 we can replace L_f by g in (4.8) and get

$$c' g \leq \mu_f^{1/t} \leq c \;\Rightarrow\; c' g^{t+\delta} \leq \mu_f^{(t+\delta)/t} \leq c g^{t+\delta}$$

since g is non–negative. We have thus found an $\varepsilon = \delta/t > 0$ such that

$$\begin{aligned}\fint_B \mu_f^{1+\varepsilon} dx \le c \fint_B g^{t+\delta} dx \le c\left(\fint_B g dx\right)^{t+\delta} \\ \le c\left(\fint_B \mu^{1/t} dx\right)^{t+\delta} \le c\left(\fint_B \mu_f dx\right)^{(t+\delta)/t}\end{aligned}$$

by the Hölder inequality. The claim follows. $\square$

Remark that as a corollary of Theorem 4.13 we get higher integrability for L_f. The assumption that the spaces are locally compact is actually redundant in this theorem. However, it is a standard assumption assuring that the spaces are somewhat reasonable.

5 Self improving property of Muckenhoupt weights

Muckenhoupt weights form a class of functions that satisfy one type of a reverse Hölder inequality. More precisely, if $1 < p < \infty$, a locally integrable non–negative function w is in A_p if for all balls B in X the inequality

$$\left(\fint_B \omega d\mu\right)\left(\fint_B w^{1-p'} d\mu\right)^{p-1} \le c_w$$

holds. The constant c_w is called the A_p–constant of w and $1/p + 1/p' = 1$. Moreover, A_1 is the class of locally integrable non–negative functions that satisfy

$$\fint_B w d\mu \le c_w \operatorname*{ess\,inf}_{x\in B} w(x).$$

for all balls B in X. In this section we show that the A_p–condition is an open ended condition; every $w \in A_p$ is also in some $A_{p-\varepsilon}$.

In the following lemma number 2 is not important and it can be replaced by any positive constant.

Proposition 5.1. *For all locally integrable non–negative functions the inequality*

$$\left(\fint_B f^{-t} d\mu\right)^{-1/t} \le \left(\fint_B f^{1/2} d\mu\right)^2 \tag{5.1}$$

holds for all $t > 0$ and all balls B in X.

Proof. Setting $g = f^{1/2}$ and replacing f by it in (5.1) gives an equivalent inequality

$$\fint_B g^{-2t} d\mu \ge \left(\fint_B g d\mu\right)^{-2t}.$$

This holds by the Jensen inequality since $x \mapsto x^{-2t}$ is a convex function on $\{x > 0\}$. $\square$

Theorem 5.2. *Let $1 \le p < \infty$ and $w \in A_p$. Then there exist a constant c and $\varepsilon > 0$ such that*

$$\left(\fint_B w^{1+\varepsilon} d\mu \right)^{1/(1+\varepsilon)} \le c \fint_B w d\mu, \tag{5.2}$$

where the constant depends only on the A_p–constant of w and on the constants in the Gehring lemma.

Proof. Since $A_1 \subset A_p$ for all $p > 1$, we can assume $p > 1$. Take an arbitrary ball B in X and $w \in A_p$ for some $p > 1$. This implies

$$\left(\fint_B w d\mu \right) \le c \left(\fint_B w^{1-p'} d\mu \right)^{1-p},$$

where the right–hand side is well defined since either $w > 0$ μ–a.e. or $w \equiv 0$. By Proposition 5.1 this implies

$$\left(\fint_B w d\mu \right) \le c \left(\fint_B w^{1/2} d\mu \right)^2. \tag{5.3}$$

Now from the Gehring lemma it follows that

$$\left(\fint_B w^{1+\epsilon} d\mu \right)^{1+\epsilon} \le c \left(\fint_B w^{1/2} d\mu \right)^2,$$

where we can use the Hölder inequality and get to

$$\left(\fint_B w^{1+\epsilon} d\mu \right)^{1+\epsilon} \le c \fint_B w d\mu \tag{5.4}$$

for some $\varepsilon > 0$ and constant c. To see this, in (5.3) replace w by an auxiliarity function g such that $w = g^2$. Then we can rewrite (5.3) as

$$\left(\fint_B g^2 d\mu \right)^{1/2} \le c \fint_B g d\mu,$$

i.e. the reverse Hölder inequality for g. Gehring's lemma provides us with $\delta > 0$ such that

$$\left(\fint_B g^{2+\delta} d\mu \right)^{1/(2+\delta)} \le c \fint_B g d\mu.$$

This leads to (5.4) with $\varepsilon = \delta/2$. □

Corollary 5.3. *Let $1 < p < \infty$ and $w \in A_p$. There exists $p_1 < p$ such that $w \in A_{p_1}$.*

Proof. Recall that $w \in A_p$ if and only if $w^{-p'/p} \in A_{p'}$. It follows from Theorem 5.2 that there are $\varepsilon > 0$ and a constant c such that

$$\left(\fint_B (w^{-p'/p})^{1+\varepsilon} d\mu \right)^{1/(1+\varepsilon)} \le c \fint_B w^{-p'/p} d\mu. \tag{5.5}$$

In addition

$$\frac{p'}{p}(1+\varepsilon) = \frac{1+\varepsilon}{p-1} = \frac{1}{p_1-1} = \frac{p_1'}{p_1},$$

where $p_1 = p/(1+\varepsilon) - 1/(1+\varepsilon) + 1$. Since $p > 1$, $p_1 < p$. The equation (5.5) can now be written as

$$\fint_B w^{-p_1'/p_1} d\mu \leq c \left(\fint_B w^{-p'/p} d\mu \right)^{1+\varepsilon}. \tag{5.6}$$

On the other hand $-p'/p = 1 - p'$ and thus the A_p condition of w implies

$$\left(\fint_B w^{-p'/p} d\mu \right)^{p/p'} \leq c \left(\fint_B w d\mu \right)^{-1}.$$

Raising this first to the power p'/p and then to $1+\varepsilon$ we get

$$\begin{aligned} \left(\fint_B w^{-p'/p} d\mu \right)^{1+\varepsilon} &\leq c \left(\fint_B w d\mu \right)^{-p'(1+\varepsilon)/p} \\ &= c \left(\fint_B w d\mu \right)^{-p_1'/p_1}. \end{aligned} \tag{5.7}$$

From (5.6) and (5.7) we finally conclude that

$$\fint_B w^{-p_1'/p_1} d\mu \leq c \left(\fint_B w d\mu \right)^{-p_1'/p_1}.$$

This means that $w \in A_{p_1}$, where $p_1 < p$. $\square$

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