Helsinki University of Technology, Institute of Mathematics, Research Reports

Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja

Espoo 2006 A500

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Outi Elina Maasalo, Anna Zatorska–Goldstein: *Stability of quasiminimizers of the p–Dirichlet integral with varying p on metric spaces*; Helsinki University of Technology, Institute of Mathematics, Research Reports A500 (2006).

Abstract: *We prove a stability result, with respect to the varying exponent p, for a family of quasiminimizers of the p–Dirichlet energy functional on a doubling metric measure space. In addition we prove global higher integrability for upper gradients of quasiminimizers with fixed boundary data, provided the boundary data belongs to a slightly better Newtonian space.*

AMS subject classifications: Primary: 49Q20, Secondary: 31C45, 49N60

Keywords: Caccioppoli inequality, capacity, doubling measure, Gehring lemma, metric space, Newtonian space, p–fatness, Poincaré inequality, quasiminimizer, stability

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ISBN 951-22-8193-7
ISSN 0784-3143

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1 Introduction

One of the most important elliptic variational problems studied in Euclidean spaces is to minimize the p–energy functional. This is equivalent to solving the p–harmonic equation. In a general metric measure space it is not clear what the counterpart for the p–harmonic equation is. However, in such a space, the variational approach to p–harmonic functions is available; it is possible to define p-harmonic functions as minimizers of the Dirichlet integral. The basic reason is that the Sobolev spaces on a metric measure space can be defined without the notion of partial derivatives; see e.g. [8], [22]. Direct methods in the calculus of variations are also available and one can prove existence results for the Dirichlet problem; see e.g. [2], [23].

A class of functions closely related to p–harmonic functions are *quasiminimizers*. A function u is called a quasiminimizer if it minimizes the Dirichlet functional up to some multiplicative constant K, that is

$$\int |Du|^p dx \leq K \int |Dv|^p dx$$

among all functions v which have the same boundary values. The notion of quasiminimizers was introduced by Giaquinta and Giusti in [7] as a tool for unified treatment of variational integrals, elliptic equations and systems, obstacle problems and quasiregular mappings. In the setting of metric spaces, the approach via quasiminimizers is particularly useful, as the Euler equation for the p–Dirichlet energy integral does not need to exist.

In recent years several papers have been published considering quasiminimizers in the setting of doubling metric measure spaces supporting a Poincaré inequality, see e.g. [4], [16], [17], [3]. All notions of metric measure spaces that appear here are explained in section 2 below. (Local) Hölder continuity for quasiminimizers has been proved by Kinnunen and Shanmugalingam [17]. In [16], Kinnunen and Martio studied nonlinear potential theory for quasiminimizers. Boundary continuity for quasiminimizers on a bounded set Ω with fixed boundary data was examined by J. Björn [4].

In the Euclidean setting different stability questions of the p–Dirichlet integral have been studied. Li and Martio examined a quasilinear elliptic operator and proved in [18] a convergence result for solutions of an obstacle problem with varying p in a bounded subset Ω of $\mathbb{R}^n$. Later they proved a similar result for a double obstacle problem, see [19]. Both cases require a measure or a capacity thickness condition on the complement of Ω.

Lindqvist considered stability of solutions to $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$, i.e. minimizers to the corresponding variational problem with varying p. The problem is solved for a bounded subset of $\mathbb{R}^n$ in [20]. In [21] Lindqvist studies stability with respect to p of the p–harmonic eigenvalue problem. Here a question on regularity of the set Ω raises.

Assume (X, μ, d) to be a complete, locally linearly convex, doubling metric measure space that supports a weak $(1, p)$–Poincaré inequality for some $p > 1$. Our main result is the following:

Theorem 1.1. *Let Ω be an open and bounded subset of X such that $X \setminus \Omega$ is of positive p–capacity and uniformly p–fat. Let $w \in N^{1,s}(\Omega)$ for some $s > p$. Assume $p = \lim_{i\to\infty} p_i$ and let $(u_i)_{i=1}^{\infty}$ be a sequence of K–quasiminimizers of the p_i–energy in Ω with boundary data w. If*

$$u_i \to u \quad \mu\text{–a.e. in } \Omega$$

then u is a K–quasiminimizer of the p–energy integral in Ω with boundary data w.

Note that since quasiminimizers do not provide unique solutions to the Dirichlet problem, in general, even if p does not vary, they may not converge. In was shown by Kinnunen and Martio that the class of (local) quasiminimizers, for p fixed, is closed under monotone convergence, provided that the limit function is bounded.

Stability requires usually some sort of higher integrability result such as the Gehring lemma. In the setting of Theorem 1.1 we prove the global higher integrability of upper gradients of quasiminimizers.

Theorem 1.2. *Let Ω be an open and bounded subset of X such that $X \setminus \Omega$ is of positive p–capacity and uniformly p–fat. Let $w \in N^{1,s}(\Omega)$ for some $s > p$.*

If $u \in N^{1,p}(\Omega)$ is a quasiminimizer of the p–energy integral in Ω with boundary data w, then there exists $0 < \delta_0 = \delta_0(p) \le s - p$ such that $g_u \in L^{p+\delta}(\Omega)$ for all $0 < \delta < \delta_0$ and

$$\left(\int_\Omega g_u^{p+\delta}\,\mathrm{d}\mu\right)^{1/(p+\delta)} \le c\left[\left(\int_\Omega g_u^{p}\,\mathrm{d}\mu\right)^{1/p} + \left(\int_\Omega g_w^{p+\delta}\,\mathrm{d}\mu\right)^{1/(p+\delta)}\right],$$

where c depends only on p and on the constants related to the space and to the domain Ω.

One standard, yet non–trivial, assumption in the metric setting is that the space satisfies a weak $(1,q)$–Poincaré inequality for some $q < p$, where p is the natural exponent associated with the studied problem. However, as shown by Keith and Zhong [12] the Poincaré inequality is a self improving property. In quite general spaces a weak $(1,p)$–Poincaré implies a weak $(1,q)$–Poincaré for some $q < p$. The same holds also for a p–fatness condition, that is a capacity thickness property of a set. We refer the reader to sections 2.1.2 and 2.1.7 respectively.

The paper is organized as follows: in section 2 we fix the general setup and we present basic facts about analytic tools used in metric setting. Most of the results are stated without proofs, in some cases we add the proof for the reader's convenience. Since there does not exist one sufficient reference, we decided to collect all needed definitions in subsection 2.1. For more details we refer to [2], [4], [5], [8], [9], [11], [13], [16], [17], [22] and [23]. The reader familiar with metric measure spaces may omit this part. Section 3 contains the proof of Theorem 1.2 and section 4 the proof of the stability result.

Acknowledgment

The results were partly obtained when the second author visited Helsinki University of Technology in January 2006. The authors would like to thank Professor Juha Kinnunen for proposing the problem and useful discussions.

The second author is partially supported by MEiN grant no 1PO3A 005 29 and by Alexander von Humboldt Foundation.

2 Preliminaries

Our notation is standard. We assume that a ball comes always with a centre and a radius, i.e. $B = B(x,r) = \{y \in X : d(x,y) < r\}$ with $0 < r < \infty$. We denote

$$u_B = \fint_B u d\mu = \frac{1}{\mu(B)} \int_B u d\mu,$$

and when there is no possibility for confusion, we denote with λB a ball with the same center as B but λ times its radius.

Throughout the paper we assume (X, μ, d) to be a complete metric space equipped with a Borel regular measure μ satisfying $0 < \mu(B) < \infty$ for all balls B of X. We will assume that the measure is *doubling*, i.e. there exists a constant $c_d > 0$ such that for every ball B in X

$$\mu(2B) \leq c_d \mu(B).$$

We refer to this property calling (X, d, μ) or briefly X a doubling metric measure space. A doubling metric measure space that is complete is always *proper*, that is its closed and bounded subsets are compact. In addition we will assume that X is a locally linearly convex space (LLC).

If not otherwise mentioned, all constants depend only on the constants of the space X, i.e. the doubling constant and the constant of the Poincaré inequality. We allow dependence on the domain Ω and on its characteristical constants that are clear in each context. Constants may also depend on the quasiminimality constant K.

2.1 Basic definitions

2.1.1 Upper gradients

Let u be a real valued function on X. A non–negative Borel measurable function g on X is said to be an *upper gradient* of u if for all rectifiable paths γ joining points x and y in X we have

$$|u(x) - u(y)| \leq \int_\gamma g ds. \tag{2.1}$$

If the above property fails only for a set of paths that is of zero p–modulus (see e.g. [11, Section 2.3] for the definition of the p–modulus of a family of paths), then g is said to be a *p–weak upper gradient* of u. We recall that if $1 < p < \infty$, every function u that has a p–integrable p–weak upper gradient has a *minimal p–integrable p–weak upper gradient* denoted g_u.

It is important to notice that for every $c \in \mathbb{R}$ the minimal p–weak upper gradient satisfies $g_u = 0$ μ–almost everywhere on the set $\{x \in X \colon u(x) = c\}$.

2.1.2 Poincaré inequality

We say that the space supports a weak $(1, q)$–Poincaré inequality if there exist $c > 0$ and $\tau \geq 1$ such that

$$\fint_B |u - u_B| \, \mathrm{d}\mu \leq cr \left(\fint_{\tau B} g^q \, \mathrm{d}\mu \right)^{1/q}$$

for all balls $B(x, r)$ in X and all pairs $\{u, g\}$ where u is a locally integrable function on X and g is a q–weak upper gradient of u. A result of [9] shows that in a doubling measure space a weak $(1, q)$–Poincaré inequality implies a weak (t, q)–Poincaré inequality for some $t > q$ and possibly a new τ i.e. there exist $c' > 0$ and $\tau' \geq 1$ such that

$$\left(\fint_B |u - u_B|^t \, \mathrm{d}\mu \right)^{1/t} \leq c' r \left(\fint_{\tau' B} g^q \, \mathrm{d}\mu \right)^{1/q}, \tag{2.2}$$

where

$$\begin{cases} 1 \leq t \leq Qq/(Q - q) & \text{if} \quad q < Q, \\ 1 \leq t & \text{if} \quad q \geq Q, \end{cases}$$

for all balls B in X, and $Q = \log_2 c_d$.

Let $1 < p < \infty$. We assume that X supports a weak $(1, p)$–Poincaré inequality. In a complete doubling metric measure space supporting a weak $(1, p)$–Poincaré inequality, there exists $1 < q < p$ such that the space admits a weak $(1, q)$–Poincaré inequality by a result in [12]. Increasing q if necessary we may additionally assume that $p \in (q, q^*)$, where $q^* = qQ/(Q - q) < \infty$.

2.1.3 Newtonian Spaces

Let $1 \leq p < \infty$. We define the space $\widetilde{N}^{1,p}(X)$ to be the collection of all p–integrable functions u on X that have a p–integrable p–weak upper gradient g on X. This space is equipped with the seminorm

$$||u||_{\widetilde{N}^{1,p}(X)} = ||u||_{L^p(X)} + \inf ||g||_{L^p(X)},$$

where the infimum is taken over all p–weak upper gradients of u. We define the equivalence relation in $\widetilde{N}^{1,p}(X)$ by saying that $u \sim v$ if

$$||u - v||_{\widetilde{N}^{1,p}(X)} = 0.$$

The *Newtonian space* $N^{1,p}(X)$ is then defined to be the space $\widetilde{N}^{1,p}(X)/\sim$ with the norm

$$||u||_{N^{1,p}(X)} = ||u||_{\widetilde{N}^{1,p}(X)}.$$

2.1.4 Capacity

The *p–capacity* of a set $E \subset X$ is defined by

$$C_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E. We say that a property holds p–quasieverywhere (p–q.e.) if the set of points for which the property fails it is of zero p–capacity.

Let Ω be a bounded subset of X and let $E \subset\subset \Omega$, i.e. E is compactly contained in Ω. We define the *relative p–capacity* of E with respect to Ω by

$$\operatorname{cap}_p(E, \Omega) = \inf_u \int_\Omega g_u^p \,\mathrm{d}\mu,$$

where the infimum is taken over all $u \in N^{1,p}(\Omega)$ such that $u \geq 1$ on E and $u = 0$ on $X \setminus \Omega$ p–quasieverywhere. Lemma 2.2 in section 2.2 shows that in a doubling metric measure space admitting a weak Poincaré inequality, the measure and the capacities are comparable.

2.1.5 Newtonian spaces with zero boundary values

Let Ω be an arbitrary subset of X. We define $N_0^{1,p}(\Omega)$ to be the set of functions $u \in N^{1,p}(\Omega)$ that are zero on $X \setminus \Omega$ p–quasieverywhere. The space $N_0^{1,p}(\Omega)$ is equipped with the norm

$$||u||_{N_0^{1,p}(\Omega)} = ||u||_{N^{1,p}(\Omega)}.$$

There are several approaches to define Newtonian spaces with zero boundary values. In general these approaches imply different spaces but it can be shown that for a wide class of metric spaces the definitions agree. Let us present two other definitions based on Lipschitz functions.

Define $\operatorname{Lip}_0^{1,p}(\Omega)$ to be the collection of all Lipschitz functions in $N^{1,p}(X)$ that vanish on $X \setminus \Omega$ and let $\operatorname{Lip}_{C,0}^{1,p}(\Omega)$ be the collection of functions in $\operatorname{Lip}_0^{1,p}(\Omega)$ that have compact support in Ω. Let $H_0^{1,p}(\Omega)$ be the closure of $\operatorname{Lip}_0^{1,p}(\Omega)$ in the norm of $N^{1,p}(X)$ and $H_{C,0}^{1,p}(\Omega)$ be the closure of $\operatorname{Lip}_{C,0}^{1,p}(\Omega)$ in the norm of $N^{1,p}(X)$. If X is proper, doubling metric measure space supporting a $(1,p)$–Poincaré inequality and Ω an open subset of X, then

$$H_{C,0}^{1,p}(\Omega) = H_0^{1,p}(\Omega) = N_0^{1,p}(\Omega).$$

The subject is discussed and the equality is proved in [23].

2.1.6 Quasiminimizers

Let Ω be an open subset of X. Let $w \in N^{1,p}(\Omega)$. We say that $u \in N^{1,p}(\Omega)$ is a quasiminimizer of the p–energy integral in Ω with boundary data w, if

$u - w \in N_0^{1,p}(\Omega)$ and there exists a constant $K > 0$ such that for all open $\Omega' \subset\subset \Omega$ and all $\phi \in N_0^{1,p}(\Omega')$ we have

$$\int_{\overline{\Omega'}} g_u^p \,\mathrm{d}\mu \le K \int_{\overline{\Omega'}} g_{u+\phi}^p \,\mathrm{d}\mu\,. \tag{2.3}$$

Quasiminimizers can be defined in several different ways. For example the integral can be taken just over Ω' instead of its closure. Also requiring that Ω' is compactly contained in Ω is not necessary. As for test functions, it is possible to use compactly supported Lipschitz functions or $\phi \in N^{1,p}(\Omega)$ such that $\operatorname{supp}\phi \subset\subset \Omega$ instead of $N_0^{1,p}(\Omega)$–functions. Also, in these cases the integral in (2.3) can be taken over the support of ϕ or the set $\{\phi \neq 0\}$. All these definitions are equivalent. For further discussion and the equivalence proof see [1].

2.1.7 LLC property and p–fatness

The local linear convexity i.e. LCC–property of X means that there exist constants $C > 0$ and $r_0 > 0$ such that for all balls B in X with radius at most r_0, every pair of points in the annulus $2B \setminus \overline{B}$ can be connected by a curve lying in the annulus $2CB \setminus C^{-1}\overline{B}$.

We say that the set $E \subset X$ is uniformly p–fat if there exist constants $c_f > 0$ and $r_0 > 0$ such that for all $x \in E$ and $0 < r < r_0$

$$\operatorname{cap}_p(E \cap B(x,r); B(x,2r)) \ge c_f \operatorname{cap}_p(B(x,r); B(x,2r)).$$

If X is a proper, LLC, doubling metric measure space supporting a $(1,q)$–Poincaré inequality for some $1 < q < p$ and Ω is an open and bounded subset of X such that $\operatorname{cap}_p(X \setminus \Omega) > 0$ and $X \setminus \Omega$ is uniformly p–fat, then Theorem 1.2 in [5] says that $X \setminus \Omega$ is also uniformly p_0–fat for some $p_0 < p$.

2.2 Preliminary results

Here we collect some basic facts concerning properties of capacity, Newtonian spaces and Sobolev–Poincaré type inequalities in the metric setting.

We start with an upper gradient lemma. Its proof follows the same way as the proof of Lemma 2.4. in [16].

Lemma 2.1. *Suppose that $u, v \in N^{1,p}(X)$ and that η is a Lipschitz continuous function in X with $0 \le \eta \le 1$. Let g_u, g_v and g_η be the p–weak upper gradients of u, v and η, respectively. Define $w = u + \eta(v - u)$. Then*

$$g_w \le (1-\eta) g_u + \eta g_v + |v-u| g_\eta$$

μ–almost everywhere in X.

The next lemma provides an estimate for the capacity of a ball and shows that capacities cap_p and C_p are essentially equivalent. For the proof see [4].

Lemma 2.2. *Let X be a doubling metric measure space admitting a weak $(1,q)$–Poincaré inequality and let $E \subset B = B(x_0, r)$ with $0 < r < \operatorname{diam} X/6$. There exists $c > 0$ such that*

$$\frac{\mu(E)}{cr^q} \le \operatorname{cap}_q(E, 2B) \le \frac{c\mu(B)}{r^q} \tag{2.4}$$

and

$$\frac{C_q(E)}{c(1+r^q)} \le \operatorname{cap}_q(E, 2B) \le 2^{q-1}\left(1 + \frac{1}{r^q}\right) C_q(E).$$

The following proposition is a capacity version of the Sobolev–Poincaré inequality. The proof is a straightforward generalization of the Euclidean case, nevertheless we present it here for the reader's convenience. One can also see [4] for a proof of the appropriate Poincaré inequality.

Proposition 2.3. *Let X be a doubling metric measure space admitting a weak $(1,q)$–Poincaré inequality and $u \in N^{1,q}(X)$ be q–quasicontinuous. Then there exists $c > 0$ such that for all balls B in X and $S = \{x \in \frac{1}{2}B \colon u(x) = 0\}$ the inequality*

$$\left(\fint_B |u|^t \,\mathrm{d}\mu\right)^{1/t} \le \left(\frac{c}{\operatorname{cap}_q(S, B)} \int_{\tau' B} g^q \,\mathrm{d}\mu\right)^{1/q} \tag{2.5}$$

holds for t and τ' are as in (2.2).

Proof. If $u_B = 0$ then the assertion follows from the (t, q)–Poincaré inequality (2.2) and (2.4). We may thus assume that $u_B = 1$. Take a Lipschitz cut–off function η such that $0 \le \eta \le 1$, $\eta \equiv 1$ on $\frac{1}{2}B$, $\operatorname{supp}\eta \subset B$ and $g_\eta \le \frac{c}{r}$. Then $\phi = -\eta(u - u_B) \in N_0^{1,p}(B)$ and $\phi = 1$ on S. Therefore

$$\operatorname{cap}_q(S, B) \le \int_B g_\phi^q \,\mathrm{d}\mu\,.$$

Since $g_\phi \le |u - u_B| g_\eta + \eta g_u$, we have

$$\operatorname{cap}_q(S, B) \le \frac{c}{r^q} \int_B |u - u_B|^q \,\mathrm{d}\mu + c \int_B g_u^q \,\mathrm{d}\mu\,.$$

The space X admits the (q, q)–Poincaré inequality, so we obtain

$$\operatorname{cap}_q(S, B) \le c \int_{\tau' B} g_u^q \,\mathrm{d}\mu,$$

and therefore

$$1 = u_B \le \left(\frac{c}{\operatorname{cap}_q(S, B)} \int_{\tau' B} g_u^q \,\mathrm{d}\mu\right)^{1/q}.$$

We can now estimate

$$\begin{aligned}\left(\fint_B |u|^t \,\mathrm{d}\mu\right)^{1/t} &\le c\left(\fint_B |u - u_B|^t \,\mathrm{d}\mu\right)^{1/t} + c u_B \\ &\le \left(\frac{c}{\operatorname{cap}_q(S, B)} \int_{\tau' B} g_u^q \,\mathrm{d}\mu\right)^{1/q},\end{aligned}$$

by the (t, q)–Poincaré inequality and (2.4). $\square$

The next lemma is a Sobolev type inequality for Newtonian functions with zero boundary values. For a proof see [4] or [17].

Lemma 2.4. *Let $1 < p < \infty$ and X be a doubling metric measure space supporting a weak $(1,q)$–Poincaré inequality for some $1 < q < p$. Moreover, let $u \in N_0^{1,p}(B)$ and the radius r of B at most* $\operatorname{diam} X/3$. *Then*

$$\left(\fint_B |u|^t d\mu\right)^{1/t} \leq cr\left(\fint_B g_u^q d\mu\right)^{1/q},$$

where t and τ' are as in (2.2).

Next we present some useful results concerning Newtonian spaces with zero boundary values. Proposition 2.5 provides a characterisation for $N_0^{1,p}$–functions by means of the Hardy inequality. Lemma 2.6 gives a sufficient condition for a sequence of $N_0^{1,p}$–functions to converge to a $N_0^{1,p}$–function. Finally, Lemma 2.7 shows that $N_0^{1,p}$ can be presented as an intersection of $N^{1,p}$ and of zero Newtonian spaces with a lower exponent. For a proof of the following proposition, see [5].

Proposition 2.5. *Let X be a proper, doubling, LLC metric measure space supporting a weak $(1,q)$–Poincaré inequality for some $1 < q < p$ and suppose that Ω is a bounded domain in X such that $X \setminus \Omega$ is uniformly p–fat. Then there is a constant $c(\Omega, p) > 0$ such that a function $u \in N^{1,p}(X)$ is in $N_0^{1,p}(\Omega)$ if and only if*

$$\int_\Omega \left(\frac{|u(x)|}{\operatorname{dist}(x, X \setminus \Omega)}\right)^p \mathrm{d}\mu \leq c\int_\Omega g_u(x)^p \,\mathrm{d}\mu\,. \tag{2.6}$$

Remark that the constant c in the above proposition formally depends on p. However, if p varies inside a bounded interval, the arguments in the proof of Proposition 2.5 show that the appropriate constants are uniformly bounded. For this reason, since in our case all exponents vary inside a bounded interval (q, q^*) we omit the dependence of the constant on p.

Lemma 2.6. *In the setting of Proposition 2.5, let $(u_i)_{i=1}^\infty$ be a bounded sequence in $N_0^{1,p}(\Omega)$. If $u_i \to u$ μ–a.e., then $u \in N_0^{1,p}(\Omega)$.*

Lemma 2.6 is formulated in [13] for (X, d, μ) doubling without further requirements and for Ω open such that $X \setminus \Omega$ satisfies a measure thickness assumption. In general a measure thickness condition is stronger than a fatness assumption. However, the lemma follows also from Proposition 2.5 and the fact that $u \in N^{1,p}(\Omega)$ is in $N_0^{1,p}(\Omega)$ if

$$\frac{|u(x)|}{\operatorname{dist}(x, X \setminus \Omega)}$$

is in $L^p(\Omega)$ for an open Ω and $1 < p < \infty$. See [5] and [13].

The assertion of the next proposition is not trivial but depends on the set Ω. Even in $\mathbb{R}^n$ some type of thickness assumption on the domain is needed, see [10]. Li and Martio show in [18] that for example p–fatness of $\mathbb{R}^n \setminus \Omega$ suffices to (2.7) to hold. The same result exists in the metric case and the proof follows from Proposition 2.5.

Proposition 2.7. *Let X be a proper, doubling, LLC metric measure space supporting a weak $(1,q)$–Poincaré inequality for some $1 < q < p$ and suppose that Ω is a bounded domain in X such that $X \setminus \Omega$ is uniformly p–fat. Then*

$$N_0^{1,p}(\Omega) = N^{1,p}(\Omega) \cap \bigcap_{s<p} N_0^{1,s}(\Omega). \tag{2.7}$$

Proof. The inclusion "$\supset$" in (2.7) in clear as Ω is bounded. It remains to prove the case "$\subset$".

Since $X \setminus \Omega$ is uniformly p–fat, it is also $(p-\varepsilon)$–fat for all $\varepsilon > 0$ small enough as discussed in section 2.1.7. Consequently,

$$\int_\Omega \left(\frac{|u|}{\operatorname{dist}(z, X \setminus \Omega)} \right)^{p-\varepsilon} \mathrm{d}\mu \leq c \int_\Omega g_u^{p-\varepsilon} \, \mathrm{d}\mu \tag{2.8}$$

for all $\varepsilon > 0$ small enough, by Proposition 2.5. We show now that (2.6) holds also for p. Indeed,

$$\begin{aligned} \int_\Omega \left(\frac{|u|}{\operatorname{dist}(z, X \setminus \Omega)} \right)^{p} \mathrm{d}\mu &= \lim_{\varepsilon\to 0} \int_\Omega \left(\frac{|u|}{\operatorname{dist}(z, X \setminus \Omega)} \right)^{p-\varepsilon} \mathrm{d}\mu \\ &\leq \lim_{\varepsilon\to 0} c \int_\Omega g_u^{p-\varepsilon} \, \mathrm{d}\mu = c \int_\Omega g_u^p \, \mathrm{d}\mu, \end{aligned}$$

and the assertion follows by Proposition 2.5. □

3 Quasiminima – higher integrability of upper gradients

The local regularity of quasiminimizers (i.e. Hölder continuity) was studied by Kinnunen and Shanmugalingam in [17]. In particular they proved the following Caccioppoli type inequality.

Theorem 3.1 (Caccioppoli inequality). *Let Ω be an open subset of X. If $u \in N^{1,p}(\Omega)$ is a quasiminimizer of the p–energy integral in Ω then there exists $c > 0$ such that for all $x \in \Omega$ and $0 < r < R$ so that $B(x,R) \subset \Omega$*

$$\int_{B(x,r)} g_u^p \, \mathrm{d}\mu \leq \frac{c}{(R-r)^p} \int_{B(x,R)} |u - u_{B(x,R)}|^p \, \mathrm{d}\mu \,. \tag{3.9}$$

We prove the global higher intergrability of upper gradients of quasiminimizers. The proof follows similar way as the Euclidean proof of Kilpeläinen and Koskela in [14] for solutions of p–harmonic type equations. The growth of integrability is achieved in a standard way by application of the Gehring lemma (its proof in the metric setting may be found for example in [24]). Remark, that the lemma holds in all doubling metric measure spaces.

Theorem 3.2 (Gehring lemma). *Let $s \in [s_0, s_1]$, where $s_0, s_1 > 1$ are fixed. Let $g \in L^s_{loc}(X)$ and $f \in L^{s_1}_{loc}(X)$ be non–negative functions. Assume that there exists constant $b > 1$ such that for every ball $B \subset \sigma B \subset X$ the following inequality*

$$\fint_B g^s \,\mathrm{d}\mu \le b\left[\left(\fint_{\sigma B} g \,\mathrm{d}\mu\right)^s + \fint_{\sigma B} f^s \,\mathrm{d}\mu\right]$$

holds for some $\sigma > 1$. Then there exists $\varepsilon_0 = \varepsilon_0(s_0, s_1, c_d, \sigma, b) > 0$ such that $g \in L^{\tilde{s}}_{loc}(X, \mu)$ for $\tilde{s} \in [s, s + \varepsilon_0)$ and moreover

$$\left(\fint_B g^{\tilde{s}} \,\mathrm{d}\mu\right)^{1/\tilde{s}} \le c\left[\left(\fint_{\sigma B} g^s \,\mathrm{d}\mu\right)^{1/s} + \left(\fint_{\sigma B} f^{\tilde{s}} \,\mathrm{d}\mu\right)^{1/\tilde{s}}\right]$$

for $c = c(s_0, s_1, c_d, \sigma, b)$.

Proof of Theorem 1.2. Recall that X is a locally linearly convex space that supports a weak $(1, q)$–Poincaré inequality for some $1 < q < p$. Since $p \in (q, q^*)$, the space supports also a weak (p, q)–Poincaré inequality (see 2.1.2). Remember also that $X \setminus \Omega$ is uniformly p–fat.

As mentioned in section 2.1.7, if $X \setminus \Omega$ is uniformly p–fat, then $X \setminus \Omega$ is also uniformly p_0–fat for some $p_0 < p$. If $p_0 < q$, we can increase it in order to have $q = p_0$ and to be able to use the (p, p_0)–Poincaré inequality. If $p_0 \ge q$ then the (p, p_0)–Poincaré inequality follows from the Hölder inequality.

Choose a ball B_0 in X such that $\Omega \subset\subset B_0 \subset 2B_0$. Fix $r > 0$ and let $B = B(x_0, r)$ be a ball such that $4\lambda B \subset 2B_0$, where λ is the multiplicative coefficient of radius in the (p, p_0)–Poincaré inequality.

If $2\lambda B \subset \Omega$ then by the Caccioppoli estimate (3.9), doubling condition and (p, p_0)–Poincaré inequality we have

$$\begin{aligned}\left(\fint_B g_u^p \,\mathrm{d}\mu\right)^{1/p} &\le \frac{c}{r}\left(\fint_{2B} |u - u_{2B}|^p \,\mathrm{d}\mu\right)^{1/p}\\ &\le c\left(\fint_{2\lambda B} g_u^{p_0} \,\mathrm{d}\mu\right)^{1/p_0}.\end{aligned} \tag{3.10}$$

Assume thus that $2\lambda B \setminus \Omega \ne \emptyset$. Choose a Lipschitz cut–off function η such that $0 \le \eta \le 1$, $\eta \equiv 1$ on B, $\operatorname{supp}\eta \subset 2B$ and $g_\eta \le \frac{c}{r}$. Then $\eta(u - w) \in N_0^{1,p}(2B \cap \Omega)$ and we may use it as a test function in (2.3). Hence

$$\int_{2B\cap\Omega} g_u^p \,\mathrm{d}\mu \le K \int_{2B\cap\Omega} g_v^p \,\mathrm{d}\mu,$$

where $v = u - \eta(u - w)$ and $g_v \le |u - w| g_\eta + (1 - \eta)(g_u + g_w) + g_w$. Therefore we obtain

$$\begin{aligned}\int_{B\cap\Omega} g_u^p \,\mathrm{d}\mu \le c &\int_{(2B\setminus B)\cap\Omega} (1 - \eta)^p g_u^p \,\mathrm{d}\mu\\ &+ c\int_{2B\cap\Omega} |u - w|^p g_\eta^p \,\mathrm{d}\mu + c\int_{2B\cap\Omega} (2 - \eta)^p g_w^p \,\mathrm{d}\mu\,.\end{aligned}$$

Adding $c\int_{B\cap\Omega} g_u^p \,\mathrm{d}\mu$ to the both sides of the inequality and dividing by $(1+c)$ implies

$$\int_{B\cap\Omega} g_u^p \,\mathrm{d}\mu \quad \le \quad \theta\int_{2B\cap\Omega} g_u^p \,\mathrm{d}\mu + \frac{\theta}{r^p}\int_{2B\cap\Omega} |u \;-\; w|^p \,\mathrm{d}\mu + \theta\int_{2B\cap\Omega} g_w^p \,\mathrm{d}\mu,$$

where $\theta = c/(1+c) < 1$. Applying a standard technical iteration lemma (see [6, lemma 3.1, ch. V]) we obtain

$$\int_{B\cap\Omega} g_u^p \,\mathrm{d}\mu \le \frac{c}{r^p}\int_{2B\cap\Omega} |u-w|^p \,\mathrm{d}\mu + c\int_{2B\cap\Omega} g_w^p \,\mathrm{d}\mu\,. \tag{3.11}$$

We will consider the integrals on the right–hand side on the larger ball $4B$. We estimate the first integral on the right–hand side using Proposition 2.3 with $q = p_0$. This gives

$$\begin{aligned}\Big(\frac{c}{r^p}\fint_{4B} |u-w|^p \,\mathrm{d}\mu\Big)^{1/p} &\le \frac{c}{r}\Big(\frac{1}{\operatorname{cap}_{p_0}(S,4B)}\int_{4\lambda B} g_{u-w}^{p_0} \,\mathrm{d}\mu\Big)^{1/p_0}\\ &\le c\Big(\frac{\mu(2B)r^{-p_0}}{\operatorname{cap}_{p_0}(S,4B)}\fint_{4\lambda B} g_{u-w}^{p_0} \,\mathrm{d}\mu\Big)^{1/p_0}\end{aligned}$$

by the doubling condition. Here the set $S = \{x \in 2B\colon u(x) = w(x)\}$. Since $u = w$ p–q.e. (and thus p_0–q.e.) in $X\setminus\Omega$ and the set $X\setminus\Omega$ is uniformly p_0–fat, we have

$$\operatorname{cap}_{p_0}(S,4B) \ge \operatorname{cap}_{p_0}(2B\setminus\Omega;4B) \ge c_f \operatorname{cap}_{p_0}(2B;4B) \ge c\mu(2B)r^{-p_0}.$$

Hence,

$$\begin{aligned}\Big(\frac{c}{r^p}\fint_{4B} |u-w|^p \,\mathrm{d}\mu\Big)^{1/p} &\le c\Big(\fint_{4\lambda B} g_{u-w}^{p_0} \,\mathrm{d}\mu\Big)^{1/p_0}\\ &= c\Big(\frac{1}{\mu(4\lambda B)}\int_{4\lambda B\cap\Omega} g_{u-w}^{p_0} \,\mathrm{d}\mu\Big)^{1/p_0},\end{aligned}$$

because $u - w = 0$ p–q.e. and thus μ–a.e. in $X\setminus\Omega$ and therefore $g_{u-w} = 0$ μ–a.e. in $X\setminus\Omega$. A simple estimation gives now

$$\begin{aligned}\Big(\frac{c}{r^p}\fint_{4B} |u-w|^p \,\mathrm{d}\mu\Big)^{1/p} &\le c\Big(\frac{1}{\mu(4\lambda B)}\int_{4\lambda B\cap\Omega} g_u^{p_0} \,\mathrm{d}\mu\Big)^{1/p_0}\\ &\quad + c\Big(\frac{1}{\mu(4\lambda B)}\int_{4\lambda B\cap\Omega} g_w^{p_0} \,\mathrm{d}\mu\Big)^{1/p_0}.\end{aligned} \tag{3.12}$$

By the Hölder inequality

$$\begin{aligned}\Big(\frac{1}{\mu(4\lambda B)}\int_{4\lambda B\cap\Omega} g_w^{p_0} \,\mathrm{d}\mu\Big)^{1/p_0} &= \Big(\fint_{4\lambda B} g_w^{p_0}\chi_{4\lambda B\cap\Omega} \,\mathrm{d}\mu\Big)^{1/p_0}\\ \le \Big(\fint_{4\lambda B} g_w^{p}\chi_{4\lambda B\cap\Omega} \,\mathrm{d}\mu\Big)^{1/p} &= \Big(\frac{1}{\mu(4\lambda B)}\int_{4\lambda B\cap\Omega} g_w^{p} \,\mathrm{d}\mu\Big)^{1/p},\end{aligned} \tag{3.13}$$

so that combining (3.11), (3.13), (3.12) and using the doubling property we obtain the inequality

$$\left(\frac{1}{\mu(B)}\int_{B\cap\Omega} g_u^p \,\mathrm{d}\mu\right)^{1/p} \le b\left(\frac{1}{\mu(4\lambda B)}\int_{4\lambda B\cap\Omega} g_u^{p_0}\,\mathrm{d}\mu\right)^{1/p_0} + c\left(\frac{1}{\mu(4\lambda B)}\int_{4\lambda B\cap\Omega} g_w^{p}\,\mathrm{d}\mu\right)^{1/p}. \quad (3.14)$$

Here the constants b and c depend only on p, Ω and on the constants associated to the structure of the space.

Set now

$$g(x) = \begin{cases} g_u^{p_0} & \text{if} \quad x \in \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

$$f(x) = \begin{cases} g_w^{p_0} & \text{if} \quad x \in \Omega, \\ 0 & \text{otherwise} \end{cases}$$

and $s = p/p_0$. The inequalities (3.10) and (3.14) imply that whenever $4\lambda B \subset 2B_0$, the following reverse Hölder inequality

$$⨍_B g^s \,\mathrm{d}\mu \le b\left(⨍_{4\lambda B} g\,\mathrm{d}\mu\right)^s + c⨍_{4\lambda B} f^s\,\mathrm{d}\mu$$

holds for $s > 1$ (p is strictly greater than p_0) and with $b = b(p)$. Applying now the Gehring lemma we obtain better integrability of g and the inequality

$$\left(⨍_B g^{\tilde{s}}\,\mathrm{d}\mu\right)^{1/\tilde{s}} \le c\left[\left(⨍_{4\lambda B} g^s\,\mathrm{d}\mu\right)^{1/s} + \left(⨍_{4\lambda B} f^{\tilde{s}}\,\mathrm{d}\mu\right)^{1/\tilde{s}}\right] \quad (3.15)$$

for $c = c(b, c_d, \lambda)$ and $\tilde{s} \in [s, s + \varepsilon_0)$, where $\varepsilon_0 = \varepsilon_0(b, c_d, \lambda)$. Since the diameter of Ω is finite we may choose a finite number of balls $B(x_j, r_j)$, $j = 1, 2, \ldots, N$, such that

$$B(x_j, 2\lambda r_j) \subset B_0 \quad \text{and} \quad \Omega \subset \bigcup_{j=1}^{N} B(x_j, r_j)$$

with fixed λ. The statement now follows by multiplying (3.15) by $\mu(4\lambda B)^{1/\tilde{s}}$ and summing over $B(x_j, r_j)$. This may require changing the constant c a bit, but the change will depend only on the doubling constant c_d and on the domain Ω. Remark that with λ and c_d fixed, the constant c will depend essentially only on p. □

4 Proof of the stability result

By a remark in section 2.1.7 we can assume that $X \setminus \Omega$ is uniformly p_0–fat. Since $p = \lim_{i\to\infty} p_i$ we can assume also that $p_i \in (q, q^*)$.

Functions u_i are supposed to be not equal to the boundary data w, i.e. we assume that there is a set of positive measure where $u_i \neq w$ μ–a.e., otherwise the result is trivial.

We start with a lemma concerning uniform higher integrability of u_i and u.

Lemma 4.1. *Let u_i and u be as in Theorem 1.1. Then there exists $\varepsilon_0 > 0$ such that*

$$u_i, u \in L^{p+\varepsilon_0}(\Omega)$$
$$g_{u_i}, g_u \in L^{p+\varepsilon_0}(\Omega)$$

and there is a subsequence such that

$$g_{u_i} \rightharpoonup g_u \quad \textit{weakly in } L^{p+\varepsilon_0}(\Omega).$$

Proof. By Theorem 1.2 for every p_i there exists $\delta_i = \delta_i(p_i)$ such that the minimal p_i–weak upper gradient g_{u_i} belongs to the space $L^{p_i+\delta_i}(\Omega)$ and

$$\begin{aligned}\Big(\int_\Omega g_{u_i}^{p_i+\delta_i}\,\mathrm{d}\mu\Big)^{1/(p_i+\delta_i)} \\ \le c_i\Big(\int_\Omega g_{u_i}^{p_i}\,\mathrm{d}\mu\Big)^{1/p_i} + c_i\Big(\int_\Omega g_w^{p_i+\delta_i}\,\mathrm{d}\mu\Big)^{1/(p_i+\delta_i)}. \quad (4.16)\end{aligned}$$

Since u_i is a quasiminimizer of the p_i–energy functional in Ω with boundary data w and thus $u_i - w \in N_0^{1,p_i}(\Omega)$, we have

$$\begin{aligned}\int_\Omega g_{u_i}^{p_i}\,\mathrm{d}\mu &\le K\int_\Omega g_w^{p_i}\,\mathrm{d}\mu \\ &\le K(\mu(\Omega))^{\delta_i/(p_i+\delta_i)}\Big(\int_\Omega g_w^{p_i+\delta_i}\,\mathrm{d}\mu\Big)^{p_i/(p_i+\delta_i)},\end{aligned}$$

and therefore

$$\Big(\int_\Omega g_{u_i}^{p_i+\delta_i}\,\mathrm{d}\mu\Big)^{1/(p_i+\delta_i)} \le c_i\Big(\int_\Omega g_w^{p_i+\delta_i}\,\mathrm{d}\mu\Big)^{1/(p_i+\delta_i)}. \quad (4.17)$$

Now remark that when $p_i \in (q, q^*)$ and $p_i \to p$ we have

$$\delta_i \ge \delta_0 = \delta_0(p) \quad \text{and} \quad c_i \le c = c(p).$$

Indeed, in order to prove (4.16) we first show that a reverse Hölder inequality

$$\begin{aligned}\Big(\frac{1}{\mu(B)}\int_{B\cap\Omega} g_{u_i}^{p_i}\,\mathrm{d}\mu\Big)^{1/p_i} \le b_i\Big(\frac{1}{\mu(\sigma B)}\int_{\sigma B\cap\Omega} g_{u_i}^{p_0}\,\mathrm{d}\mu\Big)^{1/p_0} \\ + c_i\Big(\frac{1}{\mu(\sigma B)}\int_{\sigma B\cap\Omega} g_w^{p_i}\,\mathrm{d}\mu\Big)^{1/p_i} \quad (4.18)\end{aligned}$$

holds for some $\sigma > 1$ and then we apply the Gehring lemma. The constant b_i in (4.18) depends on p_i. However, when $p_i \in (q, q^*)$, it may be chosen independently on p_i i.e. $b_i \le b$ for some $b = b(p)$. The bound will depend on p due to the fact, that we apply the $(1, p_0)$–Poincaré inequality and p_0 is chosen to be sufficiently close to p. The assertion follows because of the fact that in (4.16) δ_i is inverse proportional to b_i and c_i is comparable to b_i (see e.g. [24]).

For i sufficiently large we may assume

$$p + \varepsilon_0 \le p_i + \delta_0 \le p_i + \delta_i \le s,$$

where $\varepsilon_0 = \delta_0/2$. By this assumption and the uniform bound for c_i, applying the Hölder inequality and (4.17) we obtain

$$\begin{aligned}\Big(\int_\Omega g_{u_i}^{p+\varepsilon_0}\,\mathrm{d}\mu\Big)^{1/p+\varepsilon_0} &\le c\Big(\int_\Omega g_{u_i}^{p_i+\delta_i}\,\mathrm{d}\mu\Big)^{1/(p_i+\delta_i)}\\ &\le c\Big(\int_\Omega g_w^s\,\mathrm{d}\mu\Big)^{1/s} < \infty.\end{aligned}$$

Since

$$\begin{aligned}\Big(\int_\Omega g_{u_i-w}^{p+\varepsilon_0}\,\mathrm{d}\mu\Big)^{1/p+\varepsilon_0} &\le \Big(\int_\Omega g_{u_i}^{p+\varepsilon_0}\,\mathrm{d}\mu\Big)^{1/p+\varepsilon_0} + \Big(\int_\Omega g_w^{p+\varepsilon_0}\,\mathrm{d}\mu\Big)^{1/p+\varepsilon_0}\\ &\le c\Big(\int_\Omega g_w^s\,\mathrm{d}\mu\Big)^{1/s},\end{aligned}$$

it follows that

$$\sup_i \|g_{u_i-w}\|_{L^{p+\varepsilon_0}(\Omega)} < \infty. \tag{4.19}$$

Using Proposition 2.3 we are able to find a uniform $L^{p+\varepsilon_0}$–bound for the sequence $(u_i - w)$ as well. Observe, that decreasing ε_0 if necessary, we may additionally assume that $p + \varepsilon_0 < q^*$. So choose $t = p + \varepsilon_0$, $q = p + \varepsilon_0$ in Proposition 2.3 and fix $B_0 = B(x_0, r_0)$ such that $\Omega \subset B_0$. We note again that the minimal p_i–weak upper gradient of $u_i - w$ satisfies $g_{u_i-w} = 0$ μ-a.e. on the set $S = \{x \in B_0 \colon u(x) = w(x)\}$. On the other hand $u_i - w$ is zero p_i–quasieverywhere on $X \setminus \Omega$ and thus μ–almost everywhere on $X \setminus \Omega$. In addition, observe that p–fatness always implies $p + \varepsilon_0$–fatness, so that $\mathrm{cap}_{p+\varepsilon_0}(S, 2B_0) \ge c\mu(B_0)r^{p+\varepsilon_0}$. It follows that

$$\begin{aligned}\Big(\int_\Omega |u_i - w|^{p+\varepsilon_0}\,\mathrm{d}\mu\Big)^{1/p+\varepsilon_0} &\le c\Big(\fint_{2B_0} |u_i - w|^{p+\varepsilon_0}\,\mathrm{d}\mu\Big)^{1/p+\varepsilon_0}\\ &\le \Big(\frac{c}{\mathrm{cap}_{p+\varepsilon_0}(S, 2B_0)}\int_{2\tau' B_0} g_{u_i-w}^{p+\varepsilon_0}\,\mathrm{d}\mu\Big)^{1/p+\varepsilon_0}\\ &\le c r_0\Big(\int_\Omega g_{u_i-w}^{p+\varepsilon_0}\,\mathrm{d}\mu\Big)^{1/p+\varepsilon_0}\\ &\le c\Big(\int_\Omega g_w^s\,\mathrm{d}\mu\Big)^{1/s},\end{aligned}$$

by the Hölder inequality. Together with (4.19) this implies

$$\sup_i \|u_i - w\|_{N^{1,p+\varepsilon_0}(\Omega)} < \infty.$$

So the sequence $(u_i - w)$ is uniformly bounded in $N^{1,p+\varepsilon_0}(\Omega)$ and it follows that there exist $\tilde{u} \in N^{1,p+\varepsilon_0}(\Omega)$ and a subsequence (that we continue denoting $(u_i - w)$) such that

$$\begin{array}{ccc} u_i - w \to \tilde{u} - w & \text{in} & L^{p+\varepsilon_0}(\Omega) \\ g_{u_i} \rightharpoonup g_{\tilde{u}} & \text{weakly in} & L^{p+\varepsilon_0}(\Omega). \end{array}$$

Since $u_i \to u$ μ–a.e. it follows that $\tilde{u} = u$ μ–a.e. and equally $g_{\tilde{u}} = g_u$ μ–a.e. □

Let $D \subset \Omega$ be a compact set and for $t > 0$ write

$$D(t) = \{x \in \Omega \colon \operatorname{dist}(x, D) < t\}.$$

Then $D(t) \subset\subset \Omega$ for $t \in (0, t_0)$ where $t_0 = \operatorname{dist}(D, X \setminus \Omega)$. We reformulate a lemma by Kinnunen and Martio [16] so that it corrensponds to the present case.

Lemma 4.2. *Let u_i, u be as in Theorem 1.1. Then for almost every $t \in (0, t_0)$ we have*

$$\limsup_{i\to\infty} \int_{D(t)} g_{u_i}^{p_i} \,\mathrm{d}\mu \le c \int_{D(t)} g_u^p \,\mathrm{d}\mu,$$

where the constant c depends only on K and p.

Proof. Let $0 < t' < t < t_0$. Choose a Lipschitz cut–off function η such that $0 \le \eta \le 1$ and

$$\begin{array}{ll} \eta = 1 & \text{on} \quad D(t'), \\ \eta = 0 & \text{on} \quad \Omega \setminus D(t). \end{array}$$

Define a function

$$\phi_i = \eta(u - u_i).$$

For i large enough $p_i < p + \varepsilon_0$. Then, since u_i and u belong to $N^{1,p+\varepsilon_0}(\Omega)$ it follows that $\phi_i \in N_0^{1,p_i}(D(t))$. Therefore by the quasiminimizing property of u_i we have

$$\int_{D(t')} g_{u_i}^{p_i} \,\mathrm{d}\mu \le \int_{D(t)} g_{u_i}^{p_i} \,\mathrm{d}\mu \le K \int_{D(t)} g_{u_i+\phi_i}^{p_i} \,\mathrm{d}\mu \,.$$

Lemma 2.1 implies

$$g_{u_i+\phi_i} \le (1-\eta) g_{u_i} + g_\eta |u - u_i| + \eta g_u,$$

and hence

$$\int_{D(t')} g_{u_i}^{p_i} \,\mathrm{d}\mu \le c\bigg(\int_{D(t)} (1-\eta)^{p_i} g_{u_i}^{p_i} \,\mathrm{d}\mu \\ + \int_{D(t)} g_\eta^{p_i} |u - u_i|^{p_i} \,\mathrm{d}\mu + \int_{D(t)} \eta^{p_i} g_u^{p_i} \,\mathrm{d}\mu \bigg)$$

with c depending only on D and p_i. Observing that $\eta \equiv 1$ on $D(t')$ we add $c\int_{D(t')} g_{u_i}^{p_i}\,\mathrm{d}\mu$ to the both sides of the inequality and obtain

$$
\begin{aligned}
(1+c)\int_{D(t')} g_{u_i}^{p_i}\,\mathrm{d}\mu \le c\Bigg(&\int_{D(t)} (1-\eta)^{p_i} g_{u_i}^{p_i}\,\mathrm{d}\mu \\
&+ \int_{D(t)} g_\eta^{p_i}|u-u_i|^{p_i}\,\mathrm{d}\mu + \int_{D(t)} \eta^{p_i} g_u^{p_i}\,\mathrm{d}\mu\Bigg).
\end{aligned}
$$

Define now on $(0,t_0)$ a function

$$
\Psi(t) = \limsup_{i\to\infty} \int_{D(t)} g_{u_i}^{p_i}\,\mathrm{d}\mu\,.
$$

By definition Ψ is a nondecreasing function of t and by the uniform higher integrability of u_i it is finite for every $t \in (0,t_0)$. Therefore its set of points of discontinuity is at most countable. Let t be a point of continuity of Ψ. Taking limes superior on both sides of the last inequality we obtain

$$
(1+c)\Psi(t') \le c\Psi(t) + c\limsup_{i\to\infty}\int_{D(t)} |u-u_i|^{p_i}\,\mathrm{d}\mu + c\int_{D(t)} g_u^p\,\mathrm{d}\mu\,.
$$

The second term on the right hand side tends to zero. To see this, apply first the Hölder inequality and then the Lebesgue monotone convergence theorem. Hence, since t is a point of continuity of Ψ we obtain

$$
(1+c)\Psi(t) \le c\Psi(t) + c\int_{D(t)} g_u^p\,\mathrm{d}\mu,
$$

and furthermore

$$
\Psi(t) \le c\int_{D(t)} g_u^p\,\mathrm{d}\mu\,.
$$

□

Proof of Theorem 1.1 . In order to show that u is a quasiminimizer of the p–energy integral with boundary data w, we need to show first that $u-w \in N_0^{1,p}(\Omega)$. This does not follow immediately from the compactness argument used to extract the convergent subsequence.

We proceed as follows. For every $\varepsilon > 0$ and for i sufficiently large $p_i > p-\varepsilon$ so that $u_i - w \in N_0^{1,p-\varepsilon}(\Omega)$. By the Sobolev inequality (Lemma 2.4) we get

$$
\begin{aligned}
||u_i - w||_{N_0^{1,p-\varepsilon}(\Omega)} &\le c||g_{u_i-w}||_{L^{p-\varepsilon}(\Omega)} \\
&\le c||g_{u_i-w}||_{L^p(\Omega)},
\end{aligned}
$$

i.e. the norms of $(u_i - w)$ are uniformly bounded in $N_0^{1,p-\varepsilon}(\Omega)$.

As $X\setminus\Omega$ is uniformly p_0–fat, it is also uniformly $(p-\varepsilon)$–fat for ε small enough. In addition, $u_i \to u$ μ–a.e., so by Lemma 2.6

$$
u - w \in N_0^{1,p-\varepsilon}(\Omega)
$$

for all $\varepsilon > 0$ such that $p_0 < p - \varepsilon$. Hence, by Proposition 2.7, the p–fatness of $X \setminus \Omega$ implies also

$$u - w \in N_0^{1,p}(\Omega).$$

It remains to show that for every open $\Omega' \subset\subset \Omega$ and every $\phi \in N_0^{1,p}(\Omega')$

$$\int_{\overline{\Omega'}} g_u^p \, \mathrm{d}\mu \le K \int_{\overline{\Omega'}} g_{u+\phi}^p \, \mathrm{d}\mu \, . \tag{4.20}$$

Let $\varepsilon > 0$ be arbitrary. For i sufficiently large $p_i > p - \varepsilon$. Since (g_{u_i}) converges weakly to g_u, for every μ–measurable subset E of Ω we have

$$\begin{aligned}
\int_E g_u^{p-\varepsilon} \, \mathrm{d}\mu &\le \liminf_i \int_E g_{u_i}^{p-\varepsilon} \, \mathrm{d}\mu \\
&\le \liminf_i \left(\int_E g_{u_i}^{p_i} \, \mathrm{d}\mu \right)^{(p-\varepsilon)/p_i} \mu(E)^{1-(p-\varepsilon)/p_i} \\
&\le \liminf_i \left(\int_E g_{u_i}^{p_i} \, \mathrm{d}\mu \right)^{(p-\varepsilon)/p} \mu(E)^{\varepsilon/p},
\end{aligned}$$

where we use the Hölder inequality. Passing to zero with ε we conclude that

$$\int_E g_u^p \, \mathrm{d}\mu \le \liminf_i \int_E g_{u_i}^{p_i} \, \mathrm{d}\mu \, . \tag{4.21}$$

We will first show that the inequality (4.20) holds for every Lipschitz function compactly supported in Ω', i.e. for $\phi \in \mathrm{Lip}_C(\Omega')$. Fix $\varepsilon > 0$ and choose open sets Ω'' and Ω_0 such that

$$\Omega' \subset\subset \Omega'' \subset\subset \Omega_0 \subset\subset \Omega$$

and

$$\int_{\Omega_0 \setminus \overline{\Omega'}} g_u^p \, \mathrm{d}\mu < \varepsilon.$$

Let η be a Lipschitz cut–off function such that $0 \le \eta \le 1$, $\eta = 1$ in a neighbourhood of $\overline{\Omega'}$, and $\eta = 0$ in $\Omega \setminus \Omega''$. Define a function ϕ_i as

$$\phi_i = \phi + \eta(u - u_i).$$

Since $\phi \in \mathrm{Lip}_C(\Omega')$ and both $u_i, u \in N^{1,p+\varepsilon_0}(\Omega)$ it follows that $\phi_i \in N_0^{1,p_i}(\Omega'')$. Hence by the quasiminimizing property of u_i we get

$$\begin{aligned}
\int_{\Omega''} g_{u_i}^{p_i} \, \mathrm{d}\mu &\le K \int_{\Omega''} g_{u_i+\phi_i}^{p_i} \, \mathrm{d}\mu \\
&= K \int_{\overline{\Omega'}} g_{u_i+\phi_i}^{p_i} \, \mathrm{d}\mu + \int_{\Omega'' \setminus \overline{\Omega'}} g_{u_i+\phi_i}^{p_i} \, \mathrm{d}\mu \, .
\end{aligned} \tag{4.22}$$

Since $\eta \equiv 1$ in a neighbourhood of $\overline{\Omega'}$ it follows that

$$u_i + \phi_i = u + \phi \quad \text{in a neighbourhood of } \overline{\Omega'}. \tag{4.23}$$

On the other hand, in $\Omega'' \setminus \overline{\Omega'}$ we have $\phi \equiv 0$, and therefore

$$u_i + \phi_i = u_i + \eta(u - u_i).$$

Now Lemma 2.1 implies that

$$g_{u_i+\phi_i} \le (1-\eta) g_{u_i} + \eta g_u + g_\eta |u - u_i|.$$

Therefore

$$\begin{aligned}\int_{\Omega''\setminus\overline{\Omega'}} g_{u_i+\phi_i}^{p_i}\,\mathrm{d}\mu \le c \int_{\Omega''\setminus\overline{\Omega'}} (1-\eta)^{p_i} g_{u_i}^{p_i}\,\mathrm{d}\mu \\ + c\int_{\Omega''\setminus\overline{\Omega'}} \eta^{p_i} g_u^{p_i}\,\mathrm{d}\mu + c\int_{\Omega''\setminus\overline{\Omega'}} g_\eta^{p_i} |u-u_i|^{p_i}\,\mathrm{d}\mu\,. \end{aligned} \tag{4.24}$$

We estimate the integrals on the right–hand side separately.

Since $\eta \equiv 1$ on a neighbourhood of $\overline{\Omega'}$, there exists a compact set $D \subset \overline{\Omega''}$ such that $D \cap \overline{\Omega'} = \emptyset$ and

$$\int_{\Omega''\setminus\overline{\Omega'}} (1-\eta)^{p_i} g_{u_i}^{p_i}\,\mathrm{d}\mu \le \int_D g_{u_i}^{p_i}\,\mathrm{d}\mu\,.$$

For t sufficiently small we have $D(t) \subset \Omega_0 \setminus \overline{\Omega'}$. So we choose t such that we may apply lemma 4.2, in other words

$$\limsup_{i\to\infty} \int_{D(t)} g_{u_i}^{p_i}\,\mathrm{d}\mu \le c\int_{D(t)} g_u^p\,\mathrm{d}\mu\,.$$

Consequently

$$\begin{aligned}\limsup_{i\to\infty} \int_{\Omega''\setminus\overline{\Omega'}} (1-\eta)^{p_i} g_{u_i}^{p_i}\,\mathrm{d}\mu &\le \limsup_{i\to\infty}\int_D g_{u_i}^{p_i}\,\mathrm{d}\mu \\ &\le \limsup_{i\to\infty}\int_{D(t)} g_{u_i}^{p_i}\,\mathrm{d}\mu \le c\int_{D(t)} g_u^p\,\mathrm{d}\mu \le c\varepsilon, \end{aligned} \tag{4.25}$$

by the choice of Ω_0.

Also the second integral is arbitrarily small. Again by the choice of Ω_0 we have

$$\limsup_{i\to\infty} \int_{\Omega''\setminus\overline{\Omega'}} \eta^{p_i} g_u^{p_i}\,\mathrm{d}\mu \le \int_{\Omega''\setminus\overline{\Omega'}} g_u^p\,\mathrm{d}\mu \le \int_{\Omega_0\setminus\overline{\Omega'}} g_u^p\,\mathrm{d}\mu \le \varepsilon. \tag{4.26}$$

Observe, that for a Lipschitz function its minimal p–weak upper gradient is bounded by its Lipschitz constant μ–almost everywhere. This allows us to conclude that

$$\begin{aligned}\limsup_{i\to\infty} \int_{\Omega''\setminus\overline{\Omega'}} g_\eta^{p_i} |u-u_i|^{p_i}\,\mathrm{d}\mu &\le c \limsup_{i\to\infty}\int_{\Omega''\setminus\overline{\Omega'}} |u-u_i|^{p_i}\,\mathrm{d}\mu \\ &\le c\limsup_{i\to\infty}\left(\int_{\Omega''\setminus\overline{\Omega'}} |u-u_i|^{p+\varepsilon_0}\,\mathrm{d}\mu\right)^{\frac{p_i}{p+\varepsilon_0}} = 0, \end{aligned} \tag{4.27}$$

by the Lebesgue monotone convergence theorem.

By the estimates (4.25), (4.26) and (4.27) we obtain from (4.24)

$$\limsup_{i\to\infty} \int_{\Omega''\setminus\overline{\Omega'}} g_{u_i+\phi_i}^{p_i}\,\mathrm{d}\mu \le c\varepsilon. \tag{4.28}$$

Finally by (4.21), (4.22), (4.23) and (4.28) we have

$$\begin{aligned}\int_{\overline{\Omega'}} g_u^p\,\mathrm{d}\mu &\le \liminf_{i\to\infty} \int_{\Omega''} g_{u_i}^{p_i}\,\mathrm{d}\mu \\ &\le K \liminf_{i\to\infty} \int_{\Omega''} g_{u_i+\phi_i}^{p_i}\,\mathrm{d}\mu \\ &\le K \liminf_{i\to\infty} \int_{\overline{\Omega'}} g_{u+\phi}^{p_i}\,\mathrm{d}\mu + K \liminf_{i\to\infty} \int_{\Omega''\setminus\overline{\Omega'}} g_{u_i+\phi_i}^{p_i}\,\mathrm{d}\mu \\ &\le K \int_{\overline{\Omega'}} g_{u+\phi}^{p}\,\mathrm{d}\mu + c\varepsilon\end{aligned} \tag{4.29}$$

Passing to zero with ε we obtain the desired inequality for any $\phi \in \mathrm{Lip}_C(\Omega')$.

The result for $\phi \in N_0^{1,p}(\Omega')$ follows by approximation, i.e. if $\phi \in N_0^{1,p}(\Omega')$ then for any $\varepsilon > 0$ we may find a function $\phi_\varepsilon \in \mathrm{Lip}_C(\Omega')$ such that

$$\|\phi_\varepsilon - \phi\|_{N^{1,p}(\Omega')} < \varepsilon.$$

□

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ISBN 951-22-8193-7
ISSN 0784-3143