Helsinki University of Technology, Institute of Mathematics, Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2007 A517

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Jan Brandts Sergey Korotov Michal Křížek

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

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Jan Brandts Sergey Korotov Michal Křížek

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

Jan Brandts, Sergey Korotov, and Michal Křížek: *Simplicial finite elements in higher dimensions*; Helsinki University of Technology, Institute of Mathematics, Research Reports A517 (2007).

Abstract: *Over the past fifty years, finite element methods for the approximation of solutions of partial differential equations (PDEs) have become a powerful and reliable tool. Theoretically, these methods are not restricted to PDEs formulated on physical domains up to dimension three. Although at present there does not seem to be a very high practical demand for finite element methods that use higher dimensional simplicial partitions, there are some advantages in studying the methods independent of the dimension. For instance, it provides additional insights into the structure and essence of proofs of results in one, two and three dimensions. In this paper we review some recent progress in this direction.*

AMS subject classifications: 65N30, 51M20

Keywords: n-simplex, finite element method, superconvergence, strengthened Cauchy-Schwarz inequality, discrete maximum principle

Correspondence

brandts@science.uva.nl, sergey.korotov@hut.fi, krizek@math.cas.cz

ISBN - 978-951-22-8591-4

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, FI-02015 TKK, Finland
email:math@tkk.fi http://math.tkk.fi/

1 Motivation

The Finite Element Method (FEM) is a successful and widely applicable numerical method to approximate solutions of Partial Differential Equations (PDEs) defined on a domain $\Omega \subset \mathbb{R}^n$ [8, 13, 14]. In the FEM, $\overline{\Omega}$ is usually approximated by a face-to-face partition into simplices, after which functions that are piecewise polynomial with respect to the partition are used to approximate the solution of the PDE. One of the simplest and therefore most commonly used approximating functions are the continuous piecewise linear functions. Notice that a linear function on an n-simplex is uniquely defined by its values at the $(n + 1)$ vertices of the simplex. Therefore, specifying function values at each vertex in a face-to-face partition defines a continuous piecewise linear function.

1.1 A brief state of the art

The FEM for PDEs in two and three space dimensions is by now not only well understood, but also well coded and visualized for many different applications. Day by day, commercial software is becoming more popular and user-friendly. For instance, the software package FEMLAB [18] from COMSOL, now further developed as COMSOL Multiphysics Modelling [16], can be used by people who have only basic knowledge of the mathematical theory behind the FEM. FEMLAB can already be run on a simple PC and provides the user with easy-to-handle graphical user interfaces. Also mathematically, much progress has been made in recent years. Starting as an engineering tool, finite element theory is more and more embedded in pure mathematics, like in differential geometry. Even numerically more obscure areas in mathematics like homology theory come into play. We refer to Arnold, Falk, and Winter [3] for a good introduction into these concepts for the numerical analyst with a limited background in this area. Another recent breakthrough is the paper [29] by Stevenson who proved optimality of an adaptive finite element method for elliptic equations, which is a topic that belongs to the area of nonlinear approximation theory. Instead of a linear space of approximating functions, one employs a manifold, such as all continuous piecewise linear functions relative to any partition of a given fixed number of simplices.

1.2 Why higher dimensional finite elements?

Because of its success in two and three space dimensions, time may have come to look ahead towards finite element applications in four or even more spatial dimensions. Computational resources are rapidly becoming powerful enough to realize four-dimensional simplicial finite elements, and potential applications range from several areas in fundamental physics to financial mathematics (see [12]). Apart from that, most finite element theory has been developed independently of the spatial dimension. See for instance the papers [25, 26] which define not only the Nédélec edge- and face-elements, but in principle also define their counterparts in arbitrary space dimension. Moreover, and certainly not the least interesting reason to look at higher dimensional finite elements is that taking a bird's eye view may give further insight in the finite element method in two and three space dimensions. In fact, progress has been made by the authors of this paper in the following areas:

- Supercloseness and superconvergence
- Strengthened Cauchy-Schwarz inequalities
- Angle conditions for regularity of FEM partitions
- Assuring the discrete maximum principle
- n-section of the path-n-simplex into path-subsimplices.

The latter result, which is in the area of computational geometry, generalizes the trisection of the path-tetrahedron into three path-subtetrahedra described by Coxeter in [17]. Recall that a path n-simplex is a simplex having a path of n mutually orthogonal edges.

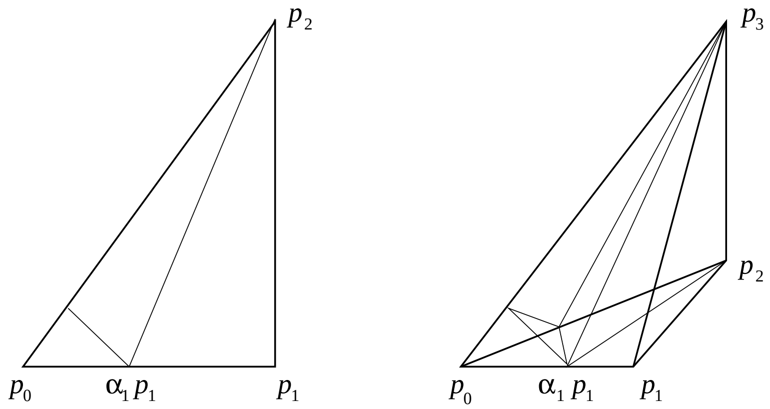

Figure 1. Cutting the path-n-simplex into $(n+1)$ path-subsimplices. The n-section is the degenerate case that results from letting α_1 tend to one.

In Figure 1, the decomposition of the right triangle into three right triangles, and of the path-tetrahedron into four path-tetrahedra is depicted. This result can be generalized to arbitrary dimension by induction.

Theorem ([11]). Each path n-simplex can be subdivided into $(n+1)$ path-subsimplices.

The dissection into only n path-subsiplices results as a degenerate case. The latter dissection can be applied recursively towards one of the two vertices that lies on the longest edge of the original simplex. This enables us to construct local refinements of simplicial partitions.

In each of the areas mentioned above, proofs have been formulated for statements independent of the spatial dimension. Although the corresponding statements in one, two, and three space dimensions were already known, their proofs in most cases look completely different for the different dimensions that are under consideration. We believe that presenting dimension independent proofs contributes to a better understanding of why the statements hold. In this paper, we aim to convince the reader that this is the case and outline the above statements. For further details, we refer to the literature.

2 The Finite Element Method

To establish notations, but also as a courtesy to the reader who is not familiar with the finite element method, we will briefly review the finite element method for elliptic partial differential equations by means of a model problem, the Poisson equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded polytopic domain with Lipschitz boundary $\partial\Omega$. Denote the space of k times continuously differentiable functions on $\overline{\Omega}$ by $C^k(\overline{\Omega})$. Given $f \in C^0(\overline{\Omega})$ we aim to find $u \in C^2(\overline{\Omega})$ such that

$$-\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega. \tag{1}$$

This is the classical formulation of the Poisson equation. We will now reformulate it such that it becomes suitable for finite element discretization.

2.1 Weak formulation

Let $v \in C_0^1(\overline{\Omega})$, where

$$C_0^1(\overline{\Omega}) = \{v \in C^1(\overline{\Omega}) \;\; | \;\; v = 0 \;\; \text{on } \partial\Omega\}. \tag{2}$$

Multiplying the first equation of (1) by v and integrating the resulting products over Ω gives, after application of Green's formula, that

$$(\nabla u, \nabla v) = (f, v), \tag{3}$$

where $(\cdot,\cdot)$ denotes the standard inner product, with associated norm, given by

$$(v, w) = \int_\Omega v \cdot w \, dx \quad \text{and} \quad \|v\|_0 = \sqrt{(v, v)}. \tag{4}$$

Here, $v \cdot w$ stands for the standard inner product between vectors, such that the same notation can be used for inner products between scalar functions and vector fields.

Conversely, consider the problem to find $u \in C_0^1(\overline{\Omega})$ such that (3) holds for all $v \in C_0^1(\overline{\Omega})$. The classical solution u of (1) clearly solves this problem. Moreover, it is easy to see that if $w \in C_0^1(\overline{\Omega})$ is another solution, then

$$(\nabla(u - w), \nabla v) = 0 \tag{5}$$

for all $v \in C_0^1(\overline{\Omega})$ and in particular for $v = u - w$, from which we conclude that $\|\nabla(u - w)\|_0 = 0$ and hence that $u = w$, since there are no non-zero constant functions in $C_0^1(\overline{\Omega})$.

If we equip $C_0^1(\overline{\Omega})$ with the norm $\|\cdot\|_1$ defined by

$$\|v\|_1 = \sqrt{\|v\|_0^2 + \|\nabla v\|_0^2}, \tag{6}$$

we find that

$$\nabla : [C_0^1(\overline{\Omega}), \|\cdot\|_1] \to [[C^0(\overline{\Omega})]^n, \|\cdot\|_0] \tag{7}$$

is a continuous mapping between normed spaces. Hence, it has a unique extension to the completions $H_0^1(\Omega)$ of $C_0^1(\overline{\Omega})$ and $(L^2(\Omega))^n$ of $(C^0(\overline{\Omega}))^n$ with respect to their norms, which is called the weak gradient. If we now consider the problem to find $u \in H_0^1(\Omega)$ such that (3) holds for all $v \in H_0^1(\Omega)$, then using the same argument as above, we see that the classical solution of (1) is the unique solution of that problem, called *the weak formulation of the Poisson equation.*

2.2 Galerkin formulation

Let V_h be a finite dimensional subspace of $H_0^1(\Omega)$ and consider the problem to find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \tag{8}$$

for all $v_h \in V_h$. This problem can be seen as an approximation of (3).

Let $v_1, \ldots, v_m$ be a basis for V_h. Then $u_h = \alpha_1 u_1 + \cdots + \alpha_m v_m$ and the coordinates $\alpha_1, \ldots, \alpha_m$ of u_h with respect to the basis can be solved from the following linear system, which can be derived from (8) using the bilinearity of inner products,

$$\begin{bmatrix} (\nabla v_1, \nabla v_1) & \ldots & (\nabla v_m, \nabla v_1) \\ \vdots & \ddots & \vdots \\ (\nabla v_1, \nabla v_m) & \ldots & (\nabla v_m, \nabla v_m) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} (f, v_1) \\ \vdots \\ (f, v_m) \end{bmatrix}. \tag{9}$$

Assume that (8), or equivalently, (9) has a solution. Then using the same arguments as above for (3), we can prove it is unique. Contrary to (3), we do not have a candidate for a solution. However, since we can easily see that choosing $f = 0$ has $u_h = 0$ as solution, and since we just argued that it is unique, we see that the so-called stiffness matrix in (9) is injective. Since it is square, it is non-singular. Thus, a unique solution exists for all $f \in C^0(\overline{\Omega})$. In fact, a unique solution exists for each f for which the right-hand side vector in (9) exists, which is for each $f \in H^{-1}(\Omega)$, the dual space of $H_0^1(\Omega)$.

2.3 Finite element approximation

Let $\mathcal{T}$ be a face-to-face partition of $\overline{\Omega}$ into simplices S, and write $\mathcal{P}^k(S)$ for the space of polynomials of degree k on S. One of the advantages of the weak formulation of the Poisson problem is, that it gives a much broader choice for the subspace V_h than the formulation (3) in $C_0^1(\overline{\Omega})$, since it can be shown that

$$V_h^k = \{v \in C^0(\overline{\Omega}): \quad v_{|S} \in \mathcal{P}^k(S) \quad \forall S \in \mathcal{T}\} \tag{10}$$

is a subspace of $H^1(\Omega)$. Since the continuous piecewise polynomials have a much simpler structure than the differentiable piecewise polynomials, this is a substantial gain. Choosing such piecewise polynomial functions leads to the Finite Element Method.

In the following, we will mostly deal with the choice $k = 1$, the continuous piecewise linear functions. A convenient basis for this space is the nodal basis, consisting of the functions from V_h^1 that have value one at exactly one vertex of the partition, and zero at all other vertices. Two convenient properties of this basis are:

- The basis functions have small support, resulting in a sparse system matrix in (9),
- The coordinates of $v \in V_h^1$ with respect to this basis are its values at the vertices.

The subscript h in V_h^1 refers to the diameter of the largest simplex in the partition with respect to which the space is defined.

3 Dimension independent results

In this section we review a number of dimension independent results.

3.1 Supercloseness and superconvergence

The continuous piecewise linear finite element approximation $u_h \in V_h^1$ of the solution u of the Poisson problem (1) resulting from (8) can be compared with other approximations of u from the same space V_h^1. Obvious candidate is the linear interpolant $L_h^1 u$, which, for u smooth enough, is the function from V_h^1 that has the same values as u at the vertices of the partition.

It was shown that, under certain conditions, the convergence to zero of the difference $\nabla u_h - \nabla L_h^1 u$ measured in the L^2-norm, is of higher order than both discrete functions converge to the exact solution u, as depicted schematically in Fig. 2. It is said that ∇u_h and $\nabla L_h^1 u$ are superclose. Notice that since u_h is the projection of u onto V_h^1 in the so-called energy inner product, the difference $u - u_h$ is on purpose depicted orthogonal to the space V_h^1. To be more explicit, supercloseness refers to results of the following type.

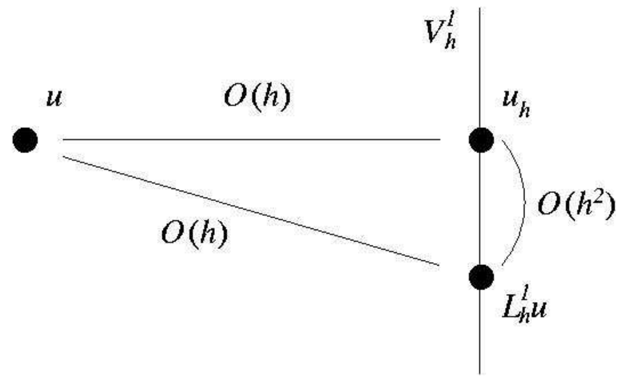

Figure 2. Superclosenesss of ∇u_h and $\nabla L_h^1 u$ when measured in the L^2-norm.

Theorem ([12]). Let $\{\mathcal{T}_h\}_{h\to 0}$ be a family of uniform partitions having the additional property of regularity, which means that there exists a constant $C > 0$ such that for all simplices S in each of the partitions we have that $\mathrm{Vol}(S) \geq Ch^n$. Then if u belongs to the Sobolev space $H^3(\Omega)$ and as h tends to zero,

$$\|\nabla(u_h - L_h^1 u)\|_0 = \mathcal{O}(h^2), \tag{11}$$

whereas only

$$\|\nabla(u - u_h)\|_0 = \mathcal{O}(h) = \|\nabla(u - L_h^1 u)\|_0.$$

The earliest reference to this result in one space dimension, in which even equality of u_h and $L_h^1 u$ occurs, is the paper [30] by Tong, although we suspect the result has been longer. In two space dimensions, the 1969 paper by Oganesjan and Ruhovets [27] is by now classical. The conditions for supercloseness in that paper are that u is three times weakly differentiable, and that each pair of triangles in the partition that share an edge, form a parallelogram. In three dimensions, the corresponding result was proved in 1980 by Chen in [15], and later by Goodsell in [20].

In the above-mentioned papers, it was not explicitly stated what the factual reason for the supercloseness was. Closer investigations of the proofs showed that there is a central property, independent of the dimension, that explains the supercloseness. This property is that if a function $v_h \in V_h^1$ is directionally differentiated along an edge, its constant derivative is the same on all simplices that share this edge. If the set of these simplices is point-symmetric with respect to its center of gravity, this leads to vanishing integrals of odd functions on the set. For an illustration, see Fig. 3. Thus, it could be proved in [12] that on simplicial partitions for which each internal edge is surrounded by such a point-symmetric patch, supercloseness occurs, provided that u is three times weakly differentiable. As a side product of the analysis, simplicial partitions of polytopes $\overline{\Omega} \subset \mathbb{R}^n$ were constructed having the desired properties.

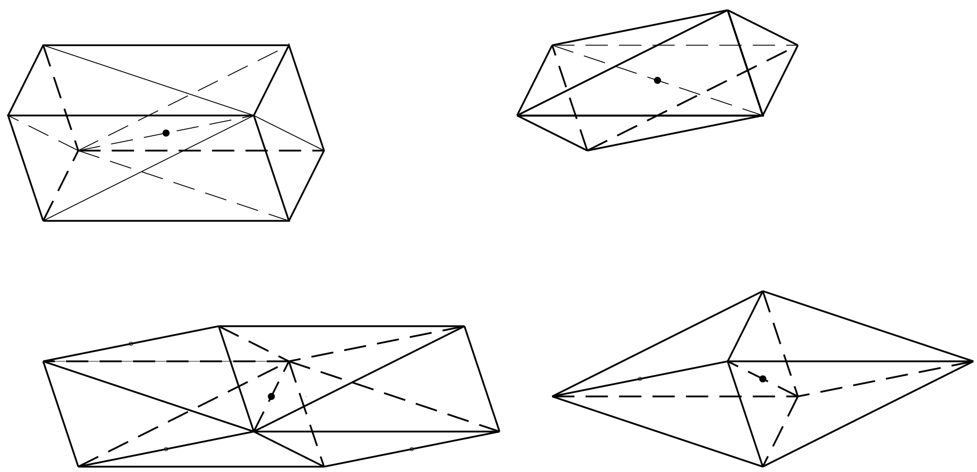

Figure 3. Point-symmetric patches in three space dimensions.

Supercloseness can be exploited as follows. The nodal interpolant $L_h^1 u$ can, since it shares values with u, be post-processed in the sense that by means of sampling at the correct points in Ω, a higher order approximation of u can be constructed. Similarly, its gradient $\nabla L_h^1 u$ can be post-processed into a vector field in $(V_h^1)^n$ that is a higher order approximation of ∇u, see [23]. Now, since ∇u_h is closer to $\nabla L_h^1 u$ than a simple triangle inequality shows, it can be proved that applying the same post-processing scheme to ∇u_h instead of to $\nabla L_h^1 u$ leads to a higher order finite element approximation of ∇u than ∇u_h itself, at a cost that is negligible compared to setting up a higher order finite element method in V_h^2, or refining the partition. This higher order approximation is then said to superconverge, and it can be used to estimate the error a posteriori. For details on superconvergence, we refer to [24] and the about one thousand references therein.

3.2 Strengthened Cauchy Schwarz inequalities

Consider a block-partitioned positive definite symmetric matrix A and its block diagonal preconditioner K,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \text{and} \quad K = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}. \tag{12}$$

It is well known that if there exists a non-negative number $\gamma < 1$ such that for all v, z of the appropriate dimensions

$$v^* A_{12} z \leq \gamma \sqrt{v^* A_{11} v} \sqrt{z^* A_{22} z}, \tag{13}$$

then the condition number $\kappa(K^{-1}A)$ of the block-diagonally preconditioned matrix $K^{-1}A$ satisfies

$$\kappa(K^{-1}A) \leq \frac{1-\gamma}{1+\gamma}, \tag{14}$$

and that the block-Jacobi iteration to approximate the solution of a linear system with system matrix A converges with the right-hand side of (14) as error reduction factor.

In the finite element method, this property is exploited as follows. Let V_h^1 be the space of continuous piecewise linear functions relative to a partition $\mathcal{T}_1$ of $\overline{\Omega} \subset \mathbb{R}^n$ that are zero on $\partial\Omega$, and W_h^1 the corresponding space relative to a refined partition $\mathcal{T}_2$ of $\overline{\Omega}$, or in other words,

$$V_h^1 \subset W_h^1, \quad \text{and} \quad W_h^1 = V_h^1 \oplus Z_h^1, \tag{15}$$

where we implicitly defined the complement space Z_h^1 of V_h^1 in W_h^1. As a basis for W_h^1 we choose the set $\mathcal{B}_1$ of nodal basis functions for V_h^1 corresponding to internal vertices of $\mathcal{T}_1$, together with the set $\mathcal{B}_2$ of nodal basis functions for W_h^1 that correspond to internal vertices in $\mathcal{T}_2$ that are not in $\mathcal{T}_1$. This naturally induces a block-partition of the finite element system matrix in (9) in which the top-left block A_{11} is the finite element matrix for the space V_h^1 only. It can be shown that inequality (13) is equivalent to the requirement

$$|(\nabla v_h, \nabla z_h)| \leq \gamma \|\nabla v_h\|_0 \|\nabla z_h\|_0, \tag{16}$$

on the coarse grid finite element space V_h^1 and its complement Z_h^1 in the fine grid space. It is easy to see that in the one-dimensional setting, this inequality holds with $\gamma = 0$.

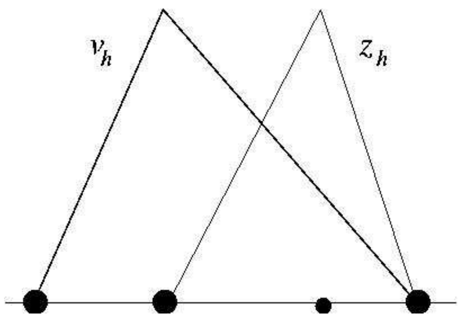

Figure 4. Orthogonality between derivatives of coarse grid basis function v_h and fine grid basis function z_h.

Indeed, as depicted in Fig. 4, the support of a nodal basis function z_h that corresponds to a fine grid vertex lies entirely in an interval I on which the derivative v_h' of the coarse grid nodal basis function v_h is constant, and thus,

$$(v_h', z_h') = v_h'|_I \int_I z_h' dx = 0. \tag{17}$$

In two space dimensions, such orthogonality does not hold, mainly because supports of fine grid basis functions stretch over two triangles on which the gradient of v_h takes different constant values. Nonetheless, in case of uniform refinement of a triangulation $\mathcal{T}_1$ into a finer triangulation $\mathcal{T}_2$, Axelsson proved in [4] that (16) holds with $\gamma = \frac{1}{2}\sqrt{2}$.

In Figure 5, uniform refinement is depicted: each triangle of the bold triangulation is subdivided into four by connecting the three midpoints of the edges of the triangle. The support of the fine grid basis function corresponding to the smaller bullet overlaps two triangles in the support of the coarse grid nodal basis function that belongs to the larger bullet.

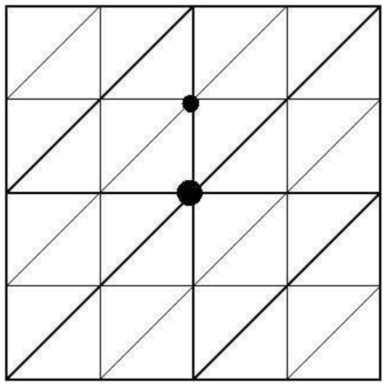

Figure 5. Uniform refinement of a triangulation and $(\nabla v_h, \nabla z_h)$ being non-trivial.

In [6], Blaheta generalized this result to tetrahedral partitions of three-dimensional domains. For this, it was necessary to define uniform refinement in three dimensions. The value for γ found there is $\gamma = \frac{1}{2}\sqrt{3}$. In the mean time, many other papers appeared on the theme of strengthened Cauchy-Schwarz inequalities, also for other types of PDEs and other FEM, see for instance [1, 2, 5, 7].

To generalize the above to arbitrary space dimensions, let $C = [0,1]^n$ be the unit n-cube. Then C can be subdivided into $n!$ simplices S of dimension n. These simplices can be characterized as the sets

$$S_\sigma = \{x \in \mathbb{R}^n \ \mid \ 0 \leq x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \leq 1\}, \tag{18}$$

where σ ranges over all $n!$ permutations of the numbers 1 to n. For $n = 3$ this results in the partition of the cube into six tetrahedra as depicted in Figure 6. Now, C can be trivially subdivided into 2^n identical subcubes, and each of the subcubes can be partitioned into $n!$ simplices using the above idea in its scaled form, resulting in a total of $n!2^n$ simplices. It can be verified that this partition also constitutes a partition of each of the $n!$ simplices S_σ from (18) in which C could have been subdivided directly; hence we have a way of subdividing the simplices of (18) into 2^n smaller ones.

By computing the singular values of certain matrices derived from the finite element matrices that belong to the coarse grid space and the fine grid space, we were able to conjecture the following value for γ_n in n space dimensions:

$$\gamma_n = \sqrt{1 - \left(\frac{1}{2}\right)^{n-1}}, \tag{19}$$

which for $n \in \{1, 2, 3\}$ corresponds to the values reported above. For each larger value of n, the statement can be directly verified by showing that γ_n

is the largest root of a real polynomial of which the coefficients are known in closed form. See [9] for details.

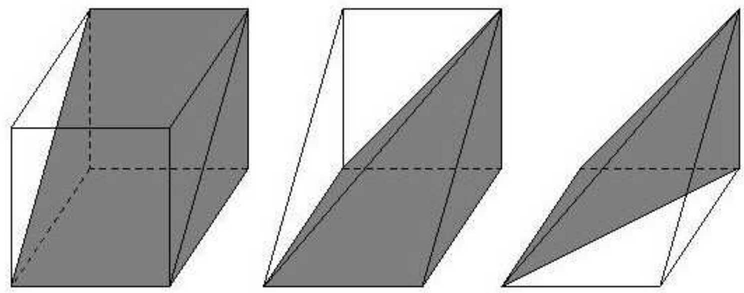

Figure 6. Partition of the cube into six tetrahedra according to (18).

3.3 Assembly of stiffness matrices and the discrete maximum principle

Let $P = (p_1|\dots|p_n)$ be a non-singular $n \times n$ matrix, and let S be the simplex with the origin p_0 and $p_1, \dots, p_n$ as vertices. Write $Q = (q_1|\dots|q_n)$ for $P^{-*} = (P^{-1})^*$, then $Q^*P = I$ shows that $q_j^* p_i = 0$ for $j \neq i$. Thus, q_j is orthogonal to the facet F_j of S opposite p_j. Since $q_j^* p_j = 1$, both p_j and q_j lie in the same half-space showing that q_j is an inward normal to F_j. Now, for $j \in \{0, \dots, n\}$, let ℓ_j be the linear function that has value one at p_j and value zero at $p_i, i \neq j$. Clearly, for $j \neq 0$ we have that

$$\ell_j : x \mapsto q_j^* x \quad \text{and} \quad q_j = \nabla \ell_j. \tag{20}$$

This leads to a natural definition of the remaining inward normal q_0 to the facet F_0 from the fact that $\ell_0 + \dots + \ell_n = 1$. Writing $e_1, \dots, e_n$ for the canonical basis vectors of $\mathbb{R}^n$, setting

$$q_0 = \nabla \ell_0 = -(q_1 + \dots + q_n) = -Qe, \quad \text{with } e = e_1 + \dots + e_n, \tag{21}$$

is consistent: since ℓ_0 vanishes on F_0, its gradient, being the direction of the strongest increase in ℓ_0, is a normal to F_0 and it points inward since $\ell_0(p_0) = 1$ is positive. Using the complete set of normals to the facets of the simplex, we can now study angle properties and the discrete maximum principle. For this, let a finite element partition $\mathcal{T}$ of $\overline{\Omega} \subset \mathbb{R}^n$ into simplices $S_1, \dots, S_\ell$ be given. Label the internal vertices of $\mathcal{T}$ by $1, \dots, m$ and let $v_1, \dots, v_m$ be the corresponding nodal basis functions. Then notice that the global stiffness matrix from (9) can be constructed as the sum

$$A = \sum_{k=1}^{\ell} A_k, \quad \text{where} \quad A_k = \begin{bmatrix} (\nabla v_1, \nabla v_1)_{S_k} & \dots & (\nabla v_m, \nabla v_1)_{S_k} \\ \vdots & \ddots & \vdots \\ (\nabla v_1, \nabla v_m)_{S_k} & \dots & (\nabla v_m, \nabla v_m)_{S_k} \end{bmatrix}, \tag{22}$$

where $(\cdot,\cdot)_{S_k}$ means that the integration takes place over S_k only. On each S_k, only the $(n+1)$ nodal basis functions that correspond to the vertices of S_k are not identically zero, showing that A_k has at most $(n+1)^2$ non-zero entries. Those entries are at the positions (i,j) in the matrix A_k with $i,j \in \{k_1,\dots,k_n\}$, where the k_j are the labels of the vertices of S_k. Now, let $\mathcal{F}_k$ be an affine invertible transform of the reference simplex $\hat{S}$ spanned by $e_1,\dots,e_n$ of $\mathbb{R}^n$ to S_k,

$$\mathcal{F}_k(x) = z_k + P_k x,$$

where $z_k \in \mathbb{R}^n$ is one of the vertices of S_k and the columns of P_k the differences of the other vertices with z_k. The $(n+1)\times(n+1)$ matrix $E_k = (e^k_{ij})$

$$e^k_{ij} = \int_{S_k} \nabla\ell_i \cdot \nabla\ell_j \, dS \quad \text{with } i,j \in \{v_1,\dots,v_{n+1}\} \tag{23}$$

is called the element stiffness matrix for the linear FEM, and its entries are equal to the entries at the positions (i,j) with $i,j \in \{k_1,\dots,k_n\}$ of A_k. From the observations above we see, with $Q_k^* P_k = I$ and q_0^k defined similar as in (21), that E_k equals

$$E_k = \left[q_0^k|Q_k\right]^* \left[q_0^k|Q_k\right] \mathrm{Vol}(S_k) = \left[q_0^k|Q_k\right]^* \left[q_0^k|Q_k\right] \frac{|\det(P_k)|}{n!}. \tag{24}$$

Thus, the stiffness matrix A in (22) is assembled from local information about the angles between the facets of the simplices $S_1,\dots,S_\ell$. Indeed, since $q_0^k,\dots,q_n^k$ are inward normals to the facets of S_k, we can define the dihedral angle between two different facets F_i^k and F_j^k of S_k as the number α_{ij}^k in $]0,\pi[$ for which

$$\alpha_{ij}^k = \pi - \gamma_{ij}^k, \tag{25}$$

where $\gamma_{ij}^k \in]0,\pi[$ is the angle between q_i^k and q_j^k. Using this, and taking the assembly of A in (22) into consideration, it is not difficult to prove that if all dihedral angles in the partition are non-obtuse (i.e., right or acute), the off-diagonal entries of A are all nonpositive. This is a sufficient condition for various discrete maximum principles to hold. See [10, 21] for details. Now, suppressing the indices k, recall that the volume of a simplex S can be computed as

$$\mathrm{Vol}(S) = \frac{h_j}{n}\mathrm{Vol}(F_j), \quad j = 1,\dots,n, \tag{26}$$

where h_j is the heigth of S above the facet F_j. This height equals the magnitude of the inner product between the vector p_j and the unit inward normal to F_j, and thus

$$h_j = \frac{p_j^* q_j}{\|q_j\|} = \frac{1}{\|q_j\|}. \tag{27}$$

Hence, by combining (24)–(27) we find a geometric interpretation of the inner product $q_i^* q_j$.

Theorem ([11]). In terms of the above notations we have that

$$q_i^* q_j = \|q_i\|\|q_j\| \cos\gamma_{ij} = -\frac{\mathrm{Vol}(F_i)\mathrm{Vol}(F_j)}{[n\mathrm{Vol}(S)]^2} \cos\alpha_{ij} \quad \text{for} \quad i \neq j \tag{28}$$

and

$$q_i^* q_i = \Big[\frac{\mathrm{Vol}(F_i)}{n\mathrm{Vol}(S)}\Big]^2.$$

This result was already derived for $n = 2$ in [19, 28] and for $n = 3$ in [22], and thus represents another example of dimension independent results. It can be compared with the following statement which is independent of angles.

Theorem ([14, p. 201]). In terms of the above notations we have that

$$\int_S v_i v_j \, dx = \frac{n!}{(n+2)!}(1 + \delta_{ij})\mathrm{Vol}(S),$$

where δ_{ij} is Kroneker's symbol.

4 Conclusions

In this paper we have argued that proving dimension-independent results in the context of the finite element method may help to gain additional insight in the statements that are proved. Therefore, instead of different proofs for different dimensions, one proof for all dimensions seems to be preferred. Examples were given in the area of superconvergence and supercloseness, strengthened Cauchy-Schwarz estimates, computation of stiffness matrices, and the discrete maximum principle.

Acknowledgments

Michak Křížek was supported by grant no. 201/04/1503 of the Grant Agency of the Czech Republic. Sergey Korotov was supported by Grant no. 112444 of the Academy of Finland. Both authors gratefully acknowledge the support.

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