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Abstract: *We show that every operator that acts between two nonseparable Hilbert spaces can be "block diagonalized", where each diagonal block acts between two separable Hilbert spaces. Analogous results hold for operator-valued* $\mathcal{H}^\infty$*,* $\mathrm{L}^\infty_{\text{strong}}$*,* $\mathcal{H}^p_{\text{strong}}$ *and* $\mathrm{L}^p_{\text{strong}}$ *functions and others.*
Using these results, several theorems about representation, interpolation, invertibility, factorization etc., which have previously been known only for separable Hilbert spaces, can now be generalized to arbitrary Hilbert spaces. We generalize several results often needed in systems and control theory, including the Nehari (or Page) Theorem, the Adamjan–Arov–Krein Theorem, the Hartman Theorem, the Lax–Halmos Theorem, Tolokonnikov's Lemma and the inner-outer factorization. We present our results both for the unit circle/disc and for the real line/half-plane.

AMS subject classifications: 47B35, 46C99, 46E40.

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1 Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary (possibly nonseparable) complex Hilbert spaces. If $\mathcal{T}$ is bounded and linear $\mathcal{X} \to \mathcal{Y}$ (i.e., $\mathcal{T} \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$), then

$$\mathcal{T} = \begin{bmatrix} * & 0 & 0 & \cdots \\ 0 & * & 0 & \cdots \\ 0 & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1}$$

where each $*$ stands for a bounded linear operator of the form $T \in \mathcal{B}(X, Y)$, where X (resp., Y) is a closed separable subspace of $\mathcal{X}$ (resp., $\mathcal{Y}$), and all such subspaces X (resp., Y) are orthogonal to each other and their (possibly uncountable) sum equals $\mathcal{X}$ (resp., $\mathcal{Y}$). The same holds if above $\mathcal{B}$ is replaced by, e.g., $\mathcal{H}^p$, $\mathcal{H}^p_{\text{strong}}$, $\mathrm{L}^p_{\text{strong}}$ or $\mathcal{H}$, which shall be defined below. Analogous claims also hold when $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is replaced by $\mathcal{B}(\mathrm{L}^2(\mathcal{X}), \mathrm{L}^2(\mathcal{Y}))$ or similar. These are the main contents of Theorem 3.2 below. Excluding holomorphicity, these results hold for real Hilbert spaces too, as explained in Theorem 7.2.

Standard interpolation results, such as the Nehari Theorem or the AAK Theorem, have been known for functions $\mathcal{T} : \mathbb{T} \to \mathcal{B}(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X}$ and $\mathcal{Y}$ are separable. Such results can be extended to general $\mathcal{X}$ and $\mathcal{Y}$ by applying the known results to each T to obtain an interpolant U and then combining all U's to a "block diagonal" function $\mathcal{U}$ that interpolates $\mathcal{T}$ in the same way. (The proof of Theorem 4.4 serves as an example.) Similar claims also hold for representation, factorization and [left] invertibility theorems. Practically the only limitation is that the interpolant, the representative, the factors or the [left] inverse must satisfy some norm estimate that does not depend on the particular subspaces involved. This condition is usually inherent in representation and interpolation results, hence nontrivial only in certain factorization and invertibility results.

In Section 2 we present our notation and introduce the space $\mathrm{L}^\infty_{\text{strong}}(\mathcal{X}, \mathcal{Y})$ of functions $F : \mathbb{T} \to \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $Fx \in \mathrm{L}^\infty(\mathcal{Y})$ for every $x \in \mathcal{X}$. It equals the set of "ℓ^2 Fourier multipliers", i.e., of functions for which the map $f \mapsto Ff$ is bounded $\mathrm{L}^2(\mathcal{X}) \to \mathrm{L}^2(\mathcal{Y})$.

In Section 3 we present the above diagonalization results in detail (some technical proofs are given in Appendix B). Due to the technicality of that section, we advice the readers interested only in Hankel and Toeplitz operators to skip Section 3 and go directly to Section 4, where we apply the results of Section 3 by generalizing the theorems mentioned in the abstract and some others to arbitrary complex Hilbert spaces. In the separable case these results are essentially known [Nik02] [Pel03] [RR85] [FF90] [Nik86].

In Section 5 we present the same results in the "continuous-time setting" where the real line and the upper half-plane take the roles of the unit circle and disc. There we also show that one can use translations instead of the shift. The results on translation-invariant subspaces have previously been known in the scalar/finite-dimensional case [Lax59], the others in the separable case.

Corresponding proofs and further details on the relations between the disc and the half-plane are presented in Section 6. There also further details and alternative results on the real line are given, with the Laguerre shift taking the role of the shift.

Naturally, our methods could be applied also to generalize similar existing separable-case results on several other groups in place of the real line or of the unit circle.

In Section 7 we establish the diagonalization method to real Hilbert spaces. This allows one to translate also much of the other sections to this real case.

Section 8 contains historical notes. Auxiliary results and some technical proofs are presented in the appendices.

The main contribution of this report is the diagonalization method of Section 3. Another contribution consists of the extension of the standard results to general Hilbert spaces in Sections 4 and 5, using the diagonalization method.

The third contribution of this report is the illustration of some pathological phenomena of $\mathrm{L}^\infty_{\mathrm{strong}}$, both within the main text and in Appendix C. Part of these appear in the separable case too. Also other $\mathrm{L}^p_{\mathrm{strong}}$ and $\mathcal{H}^p_{\mathrm{strong}}$ spaces are studied.

This report is a supplement to [Mik07a], which is recommended as an introduction containing the main ideas in a more accessible form. This report extends the results and methods of [Mik07a] and adds details and further results but is therefore more technical at the cost of readability, particularly in the treatment of the diagonalization method.

Auxiliary results that are assumed to be known by the reader can mostly be found in the appendices of [Mik02].

2 Notation and $\mathrm{L}^\infty_{\mathrm{strong}}$

In this section we present our (standard) notation. In the "continuous-time" sections 5 and 6 the notation is slightly different with, e.g., $\mathbb{R}$ in place of $\mathbb{T}$. We also present some properties of $\mathcal{H}^p$, L^p, $\mathcal{H}^p_{\mathrm{strong}}$ and $\mathrm{L}^p_{\mathrm{strong}}$.

First we recall that a Hilbert space is isomorphic to $\ell^2(W)$, where W is its orthonormal basis. The space is nonseparable iff W is uncountable. An example of a nonseparable Hilbert space is the Besicovich space (the completion of the space of almost-periodic functions; it is equivalent to $\ell^2(\mathbb{R})$).

Measurable means Bochner-measurable. We set $\|f\|_B = \infty$ when B is a Banach space and $f \notin B$. By $f[A]$ we denote the image $\{f(a) \bigm| a \in A\}$ of A. We set $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{N} := \{0, 1, 2, \ldots\}$, $\mathbb{T} := \{z \in \mathbb{C} \bigm| |z| = 1\}$, $\mathbb{D} := \{z \in \mathbb{C} \bigm| |z| < 1\}$. By $[F]$ (or by F when there is no risk of ambiguity) we denote the equivalence class of a function F (in, e.g., L^p or in $\mathrm{L}^p_{\mathrm{strong}}$). By M_F we denote the multiplication operator $f \mapsto Ff$ (i.e., $(M_F f)(z) := F(z)f(z)$), for $f \in \mathrm{L}^2$ unless some other space has been specified.

The symbols of the form $V(A;B)$ usually stand for spaces of functions $A \to B$. When $A = \mathbb{T}$ or $A = \mathbb{D}$, we often omit "A;"; when also $B = \mathcal{B}(\mathcal{X},\mathcal{Y})$, we often write $V(\mathcal{X},\mathcal{Y})$ instead of $V(A;\mathcal{B}(\mathcal{X},\mathcal{Y}))$. E.g., by $\mathrm{L}^\infty(B)$ we denote the space of (equivalence classes of) essentially bounded measurable functions $\mathbb{T} \to B$, when B is a Banach space. When $1 \le p < \infty$, we define $\mathrm{L}^p(B)$ by setting

$$\|f\|_p^p := \|f\|_{\mathrm{L}^p(B)}^p := \frac{1}{2\pi}\int_0^{2\pi} \|f(\mathrm{e}^{it})\|_B^p \, dt \tag{2}$$

Let $1 \le p \le \infty$. We denote by $\mathrm{L}^p_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ the space of (equivalence classes of) functions $F : \mathbb{T} \to \mathcal{B}(\mathcal{X},\mathcal{Y})$ for which $Fx \in \mathrm{L}^p(\mathcal{Y})$ for each $x \in \mathcal{X}$, with the norm

$$\|F\|_{\mathrm{L}^p_{\mathrm{strong}}} := \sup\{\|Fx\|_{\mathrm{L}^p(\mathcal{Y})} \,\big|\, \|x\|_{\mathcal{X}} \le 1\}. \tag{3}$$

By $\mathcal{H}^p(B)$ we denote the holomorphic functions $\mathbb{D} \to B$, where $\mathbb{D} := \{z \in \mathbb{C} \,\big|\, |z| < 1\}$ is the unit disc, with the norm

$$\|f\|_{\mathcal{H}^p} := \sup_{r<1} \|f(r\cdot)\|_{\mathrm{L}^p} < \infty. \tag{4}$$

By $\mathcal{H}^p_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ we denote the functions $F : \mathbb{D} \to \mathcal{B}(\mathcal{X},\mathcal{Y})$ for which $Fx \in \mathcal{H}^p(\mathcal{Y})$ for each $x \in \mathcal{X}$. It follows that F is holomorphic [HP57, Theorem 3.10.1] and

$$\|F\|_{\mathcal{H}^p_{\mathrm{strong}}} := \sup\{\|Fx\|_{\mathcal{H}^p(\mathcal{Y})} \,\big|\, \|x\|_{\mathcal{X}} \le 1\} < \infty. \tag{5}$$

The spaces $\mathrm{L}^p(B)$, $\mathcal{H}^p(B)$, $\mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ and $\mathcal{H}^p_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ are Banach spaces, and $\mathcal{H}^\infty_{\mathrm{strong}} = \mathcal{H}^\infty$ (by the Uniform Boundedness Theorem). However, $\mathrm{L}^p_{\mathrm{strong}}$ is an incomplete subspace of $\mathcal{B}(\mathcal{X},\mathrm{L}^p(\mathcal{Y}))$ whenever $\mathcal{X}$ and $\mathcal{Y}$ are infinite-dimensional and $p < \infty$ [Mik08, below Theorem 2.5] [Mik06a, Example 4.3].

We mention below some basic properties of $\mathrm{L}^\infty_{\mathrm{strong}}$.

Remarks 2.1 If $\dim \mathcal{X} < \infty$, then $\mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y}) = \mathrm{L}^\infty(\mathcal{B}(\mathcal{X},\mathcal{Y}))$ isometrically. Even in the infinite-dimensional case, $\mathrm{L}^\infty(\mathcal{B}(\mathcal{X},\mathcal{Y}))$ is a closed subspace of $\mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$, i.e., the two norms coincide on L^∞ (Lemma A.8). Nevertheless, in the nonseparable case we may have a (non-Bochner-measurable) function $F : \mathbb{T} \to \mathcal{B}(\mathcal{X},\mathcal{Y})$ with $\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = 0$ (i.e., $[F] = [0]$) even though $\|F(z)\|_{\mathcal{B}(\mathcal{X},\mathcal{Y})} = 1$ for each $z \in \mathbb{T}$, as shown in Example C.2 below.

If $\mathcal{X}$ and $\mathcal{Y}$ are separable, then $\mathrm{L}^\infty_{\mathrm{strong}}$ coincides with the space of essentially bounded weakly measurable functions $\mathbb{T} \to \mathcal{B}(\mathcal{X},\mathcal{Y})$, by Lemma A.8. The latter description is often used as the definition of "L^∞", (actually, of $\mathrm{L}^\infty_{\mathrm{strong}} = \mathrm{L}^\infty_{\mathrm{weak}} \supsetneq \mathrm{L}^\infty$) in the separable case; see, e.g., p. 66 of [Pel03] or pp. 81–82 of [RR85].

By [Mik08] (and Lemma 6.1), the space $\mathrm{L}^\infty_{\mathrm{strong}}$ is exactly the space of "Fourier multipliers" $\ell^2(\mathbb{Z};\mathcal{X}) \to \ell^2(\mathbb{Z};\mathcal{Y})$, where $\ell^2(\mathbb{Z};\mathcal{X})$ denotes the space of square-summable functions $\mathbb{Z} \to \mathcal{X}$. This means that bounded linear maps $\mathscr{E} : \ell^2(\mathbb{Z};\mathcal{X}) \to \ell^2(\mathbb{Z};\mathcal{Y})$ satisfying $\mathscr{E}S = S\mathscr{E}$, where $(Sf)_n = f_{n-1}$, are exactly the maps of the form $\mathscr{E} = Z^{-1}M_F Z$, where $F \in \mathrm{L}^\infty_{\mathrm{strong}}$ and $Z : \ell^2 \to \mathrm{L}^2$ is the isometric isomorphism $\ell^2 \ni g \mapsto \sum_k z^k g_k$ (cf. (21)), by Theorem 4.3. Analogously, the space of Fourier multipliers $\mathrm{L}^2(\mathbb{R};\mathcal{X}) \to \mathrm{L}^2(\mathbb{R};\mathcal{Y})$ is $\mathrm{L}^\infty_{\mathrm{strong}}(\mathbb{R};\mathcal{B}(\mathcal{X},\mathcal{Y}))$ (Theorem 5.2). $\triangleleft$

Now we recall [Mik08, Theorem 2.5] (through Lemma 6.1), which shows that any bounded linear operator $\mathcal{X} \to \mathrm{L}^\infty(\mathcal{Y})$ is determined by a $\mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ function (class) and that any $\mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ function can be redefined so as not to exceed its norm:

Proposition 2.2 *We have* $\mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y}) = \mathcal{B}(\mathcal{X},\mathrm{L}^\infty(\mathcal{Y}))$*, isometrically. Moreover, for each* $T \in \mathcal{B}(\mathcal{X},\mathrm{L}^\infty(\mathcal{Y}))$*, there exists* $F : \mathbb{T} \to \mathcal{B}(\mathcal{X},\mathcal{Y})$ *such that* $T_F : x \mapsto [Fx]$ *equals* T *and* $\sup_{\mathbb{T}} \|F\|_{\mathcal{B}(\mathcal{X},\mathcal{Y})} = \|T\|$*.*

Note that if $[F], [\tilde F] \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$, then $T_{\tilde F} = T_F$ iff $\|F - \tilde F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = 0$, although we may have $\operatorname{ess\,sup} \|\tilde F\|_{\mathcal{B}(\mathcal{X},\mathcal{Y})} = \infty$ when $\mathcal{X}$ is nonseparable, by Example C.2(c).

The *Poisson integral* of f is defined by

$$f(r\mathrm{e}^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} f(\mathrm{e}^{it})\,dt \qquad (r \ge 0,\ \theta \in [0,2\pi)). \quad (6)$$

Any $\mathcal{H}^p(\mathcal{X})$ function is the Poisson integral of an $\mathrm{L}^p(\mathcal{X})$ function:

Proposition 2.3 ($\mathcal{H}^p \subset \mathrm{L}^p$) *Let* $f \in \mathcal{H}^p(\mathcal{X})$*,* $1 \le p \le \infty$*. Then* f *has a* boundary function $f_0 \in \mathrm{L}^p(\mathcal{X})$ *such that* $f(rz) \to f_0(z)$ *for a.e.* $z \in \mathbb{T}$*, as* $r \to 1-$*. Moreover,* $\|f_0\|_p = \|f\|_{\mathcal{H}^p} = \lim_{r\to 1-} \|f(r\cdot)\|_p$*, and* f *is the Poisson integral of* f_0*. If* $p < \infty$*, then* $\|f(r\cdot) - f_0\|_p \to 0$ *as* $r \to 1-$*.*

Proof: Since $\mathbb{D}$ is separable, so is $f[\mathbb{D}]$. Therefore, we may assume that $\mathcal{X}$ is separable. Consequently, the proposition follows from [RR85, pp. 84 & 88–89]. $\square$

By $\mathcal{H}^\infty(\mathcal{X},\mathcal{Y})$ (resp., $\mathcal{H}^\infty_-(\mathcal{X},\mathcal{Y})$) we denote the Banach space of bounded holomorphic functions $\mathbb{D} \to \mathcal{B}(\mathcal{X},\mathcal{Y})$ (resp., $\mathbb{D}^- \to \mathcal{B}(\mathcal{X},\mathcal{Y})$) with the supremum norm, where $\mathbb{D}^- := \{z \in \mathbb{C} \mid |z| > 1\}$. It is a closed subspace of $\mathrm{L}^\infty_{\mathrm{strong}}$:

Proposition 2.4 ($\mathcal{H}^\infty \subset \mathrm{L}^\infty_{\mathrm{strong}}$) *Let* $F \in \mathcal{H}^\infty(\mathcal{X},\mathcal{Y})$*. Then there exists a unique* boundary function $[F_0] \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ *such that for each* $x \in \mathcal{X}$ *there is a null set* $N_x \subset \mathbb{T}$ *for which* $F(rz)x \to F_0(z)x$ *in* $\mathcal{Y}$*, as* $r \to 1-$*, for each* $z \in \mathbb{T} \setminus N_x$*.*

If $f \in \mathcal{H}^p(\mathcal{X})$ *and* $G \in \mathcal{H}^\infty(\mathcal{Y},\mathcal{Z})$ *(*$1 \le p \le \infty$*), where also* $\mathcal{Z}$ *is a Hilbert space, with boundary functions* f_0 *and* G_0*, respectively, then the boundary functions of* Ff *and* GF *equal* $F_0 f_0$ *and* $G_0 F_0$*, respectively.*

(This follows from [Mik08, Theorem 1.5] through Lemma 6.1.)

We have $\|F(rz) - F(z)\|_{\mathcal{B}(\mathcal{X},\mathcal{Y})} \to 0$ for a.e. $z \in \mathbb{T}$ iff $F_0 \in \mathrm{L}^\infty(\mathcal{B}(\mathcal{X},\mathcal{Y}))$, or equivalently, iff F_0 is Bochner-measurable (use the Poisson integral formula or Lemma A.8). Nevertheless, Fx is the Poisson integral of $F_0 x$ for each

$x \in \mathcal{X}$. Similarly, any Nevanlinna function $\mathbb{D} \to \mathcal{B}(\mathcal{X}, \mathcal{Y})$ has a boundary function (and all these boundary functions are also nontangential) [Mik08].

We identify a function $F \in \mathcal{H}^\infty(\mathcal{X}, \mathcal{Y})$ (or $F \in \mathcal{H}^p(\mathcal{X})$) by its boundary function (equivalence class) F_0, thus extending it to the boundary $\mathbb{T}$, even though the extension is unique only as a class. Therefore, $\mathcal{H}^p(\mathcal{X})$ (resp., $\mathcal{H}^\infty(\mathcal{X}, \mathcal{Y})$) is considered as a subspace of $\mathrm{L}^p(\mathcal{X})$ (resp., of $\mathrm{L}^\infty_{\text{strong}}(\mathcal{X}, \mathcal{Y})$). Note that $\mathrm{L}^2(\mathcal{X})$ and $\mathcal{H}^2(\mathcal{X})$ are Hilbert spaces. We consider $\mathcal{B}$ as the subspace of constant functions (in L^p, $\mathrm{L}^p_{\text{strong}}$, $\mathcal{H}^p$, $\mathcal{H}^p_{\text{strong}}$ or similar).

In Section 3 we define P_X, $\tilde{P}_X$, $F_{X,Y}$, $\mathcal{V}$ and F^* (for $F \in \mathrm{L}^\infty_{\text{strong}}$), in Section 4 we define H^2_-, P_+, P_-, Γ_F, S, S^* and N_F, and in Section 5 we define $\mathbb{C}^+$, τ^t, TI and TIC (and redefine $\mathcal{H}^p$, L^p, $\mathcal{H}^p_{\text{strong}}$, $\mathrm{L}^p_{\text{strong}}$, P_+, P_-, $\Gamma_{\mathscr{E}}$, $\widehat{f}$, $\mathcal{F}$ etc. for Sections 5–6).

3 Diagonalization

In Theorem 3.2 we shall present in detail the diagonalization process explained around equation (1). Before that, in Theorem 3.1, we show how to combine such "diagonal blocks" "T" to an operator $\mathcal{T}$. Some related questions and $\mathrm{L}^\infty_{\text{strong}}$ adjoints are treated later in this section.

These require some technical considerations on $\mathrm{L}^\infty_{\text{strong}}(\mathcal{X}, \mathcal{Y})$. To avoid them, one could rewrite the corresponding proofs in "time-domain", i.e., for shift-invariant operators $\ell^2(\mathbb{Z}; \mathcal{X}) \to \ell^2(\mathbb{Z}; \mathcal{Z})$ (cf. Section 5 or [Mik02, Chapter 13]). However, we prefer to write the proofs directly for $\mathrm{L}^\infty_{\text{strong}}$ and to present its properties in detail, because this is the more popular approach in the literature (in the separable case).

Recall first that if the vectors $x_\alpha \in \mathcal{X}$ are orthogonal for each $\alpha \in \mathcal{A}$, then $x := \sum_{\alpha \in \mathcal{A}} x_\alpha$ converges in $\mathcal{X}$ iff $R := \sum_{\alpha \in \mathcal{A}} \|x_\alpha\|^2 < \infty$ (in particular, at most countably many x_α may be nonzero). If $R < \infty$, then $\|x\|^2 = R$. [Rud74, Theorem 12.6]

If X (resp., Y) is a closed subspace of $\mathcal{X}$ (resp., $\mathcal{Y}$), then we denote the orthogonal projection $\mathcal{X} \to X$ by P_X (resp., $\mathcal{Y} \to Y$ by P_Y). Thus, $P_Y^* \in \mathcal{B}(Y, \mathcal{Y})$ is the canonical isometric embedding $Y \to \mathcal{Y}$. By $\tilde{P}_X = \tilde{P}_X^* \in \mathcal{B}(\mathcal{X})$ we denote the zero extension of P_X (similarly for P_Y).

We go on with some rather obvious facts on "diagonal" operators (cf. (1)):

Theorem 3.1 ($\{F_{X,Y}\} \mapsto F$) *Let $\mathcal{V}$ a collection of pairs (X,Y), where the spaces X (resp., Y) are pairwise orthogonal closed subspaces of $\mathcal{X}$ (resp., $\mathcal{Y}$).*

If $F_{X,Y} \in \mathcal{B}(X,Y)$ for all $(X,Y) \in \mathcal{V}$ and $M := \sup_{(X,Y)\in\mathcal{V}} \|F_{X,Y}\|_{\mathcal{B}(X,Y)} < \infty$, then $F := \sum_{(X,Y)\in\mathcal{V}} P_Y^* F_{X,Y} P_X$ satisfies[1]

$$F \in \mathcal{B}(\mathcal{X},\mathcal{Y}), \qquad \|F\|_{\mathcal{B}(\mathcal{X},\mathcal{Y})} = M, \tag{7}$$

$$P_Y^* F_{X,Y} P_X = \tilde{P}_Y^* F \tilde{P}_X = \tilde{P}_Y^* F = F \tilde{P}_X \tag{8}$$

$$F^* = \sum_{(X,Y)\in\mathcal{V}} P_X^* F_{X,Y}^* P_Y, \qquad \langle y, Fx\rangle_{\mathcal{Y}} = \sum_{(X,Y)\in\mathcal{V}} \langle \tilde{P}_Y y, F\tilde{P}_X x\rangle_Y \tag{9}$$

for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$. Moreover, the map $(F_{X,Y})_{(X,Y)\in\mathcal{V}} \mapsto F$ is linear.

(a1) *If $\sum_{(X,Y)\in\mathcal{V}} X = \mathcal{X}$, and $g \in \mathcal{X}$ or $g \in \mathrm{L}^2(\mathcal{X})$, then $g = \sum \tilde{P}_X g$, $\|g\|^2 = \sum \|P_X g\|^2$, and $Fg = \sum P_Y^* F_{X,Y} P_X g$. In particular, then $P_X g = 0$ (a.e.) for all but (at most) countably many $(X,Y) \in \mathcal{V}$. Conversely, if $g_X \in \mathrm{L}^2(X)$ for each $(X,Y) \in \mathcal{V}$ and $R := \sum \|g_X\|_2^2 < \infty$, then $g := \sum P_X^* g_X \in \mathrm{L}^2(\mathcal{X})$ and $\|g\|_2^2 = R$.*

(a2) *If also $\mathcal{Z}$ is a Hilbert space, the spaces Z_Y are pairwise orthogonal closed subspaces of $\mathcal{Z}$ and $G_{Y,Z_Y} \in \mathcal{B}(Y,Z)$ for each $(X,Y) \in \mathcal{V}$, and $\sup_{(X,Y)\in\mathcal{V}} \|G_{Y,Z_Y}\| < \infty$, then $G := \sum P_{Z_Y}^* G_{Y,Z_Y} P_Y \in \mathcal{B}(\mathcal{Y},\mathcal{Z})$ and $GF = \sum P_{Z_Y}^* G_{Y,Z_Y} F_{X,Y} P_X$.*

(a3) *If $\tilde{X} \subset \mathcal{X}$ is separable, then $\tilde{X}$ is contained in the direct sum of a countable subset of $\mathcal{V}^{\mathcal{X}} := \{X \mid (X,Y) \in \mathcal{V} \text{ for some } Y \subset \mathcal{Y}\}$; i.e., $\tilde{X} \perp X$ for the remaining $X \in \mathcal{V}^{\mathcal{X}}$.*

(b1) *All of the above in this theorem also holds with $\mathrm{L}^\infty_{\text{strong}}$, $\mathcal{H}^\infty$ or $\mathcal{H}^\infty_-$ in place of $\mathcal{B}$*

In the $\mathcal{H}^\infty$ (resp., $\mathcal{H}^\infty_-$) case, the sum $F(z)x := \sum P_Y^* F_{X,Y}(z) P_X x$ converges for each $z \in \mathbb{D}$ (resp., $\mathbb{D}^-$) and $x \in \mathcal{X}$; similarly, (9) may be interpreted pointwise (i.e., then F^* denotes the function $z \mapsto F(z)^*$).

However, in the $\mathrm{L}^\infty_{\text{strong}}$ case the sum converges in general just strongly in $\mathcal{B}(\mathcal{X}, \mathrm{L}^\infty(\mathcal{Y}))$ (i.e., $Fx := \sum P_Y^* F_{X,Y} P_X x$ (in $\mathrm{L}^\infty(\mathcal{Y})$) for each $x \in \mathcal{X}$). Nevertheless, the resulting map $F \in \mathcal{B}(\mathcal{X}, \mathrm{L}^\infty(\mathcal{Y}))$ is independent of the representatives $F_{X,Y}$ chosen.

Moreover, if we require that the representatives $F_{X,Y}$ of their classes are chosen so that $\sup_{t\in\mathbb{T}} \|F_{X,Y}(t)\|_{\mathcal{B}(X,Y)} \le M$ for each $(X,Y) \in \mathcal{V}$ (see Proposition 2.4), then $G(z)x := \sum P_Y^* F_{X,Y}(z) P_X x$ converges at each $z \in \mathbb{T}$, we have $\sup_{t\in\mathbb{T}} \|G(t)\|_{\mathcal{B}(\mathcal{X},\mathcal{Y})} = M$, and G is a representative of the class of F. If we let F denote this function G, then also (9) holds pointwise (in $\mathcal{B}(\mathcal{X},\mathcal{Y})$) everywhere on $\mathbb{T}$, and $M_{F^*} = M_F^*$. (For certain other representatives of the classes of each $F_{X,Y}$ the first equation in (9) need not be meaningful pointwise, merely in "$\mathrm{L}^\infty_{\text{weak}}$". See also Corollary 3.5).

(b2) *For $\mathcal{H}^2_{\text{strong}}$ in place of $\mathcal{B}$, all of the above (except (b1)) holds if in (a1) we require that $g \in \mathcal{X}$ and in (a2) we require that $\|G_{Y,Z_Y}\|_{\mathrm{L}^\infty_{\text{strong}}} \le M'$ for each $(X,Y) \in \mathcal{V}$.*

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[1]The definition means that $Fx := \sum P_Y^* F_{X,Y} P_X x$ for each $x \in \mathcal{X}$.

Above and below all sums run over $\mathcal{V}$. By $\mathcal{X} = \sum X$ in (a1) above we mean that $x = \sum P_X x$ for each $x \in \mathcal{X}$ (so $\sum \|P_X x\|^2 = \|x\|^2 < \infty$), equivalently, that $\cap_{(X,Y)\in\mathcal{V}} X^\perp = \{0\}$.

In the theorem, (particularly) in case of $\mathrm{L}^\infty_{\mathrm{strong}}$ or $\mathrm{L}^2_{\mathrm{strong}}$, the claims in (9) are meant (only) pointwise. However, in case of $\mathrm{L}^\infty_{\mathrm{strong}}$, they also hold for the "$\mathrm{L}^\infty_{\mathrm{strong}}$ adjoint", as explained in (b1).

If $F \in \mathrm{L}^2_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ and $G \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{Y},\mathcal{Z})$, then

$$\|GFx\|_2 \le \|G\|_{\mathrm{L}^\infty_{\mathrm{strong}}}\|Fx\|_2 \le \|G\|_{\mathrm{L}^\infty_{\mathrm{strong}}}\|F\|_{\mathrm{L}^2_{\mathrm{strong}}}\|x\|_{\mathcal{X}} \quad (x \in \mathcal{X}), \tag{10}$$

by [Mik08, Lemma 2.2]. This was applied in (b2)&(a2) above. The proof of (b2) would also cover the case $\mathrm{L}^2_{\mathrm{strong}}$ except that we could only guarantee that $F \in \mathcal{B}(\mathcal{X},\mathrm{L}^2(\mathcal{Y}))$, not that F has a representative of the form $\mathbb{T} \to \mathcal{B}(\mathcal{X},\mathcal{Y})$.

Proof of Theorem 3.1: We start by proving all but (a1), (a2), (a3) and (b2) (i.e., just the initial claims for the cases of $\mathcal{B}$, $\mathcal{H}^\infty$, $\mathcal{H}^\infty_-$ and $\mathrm{L}^\infty_{\mathrm{strong}}$). Note first that the linearity claim is obvious in all settings. Without loss of generality, we assume that $\sum_{(X,Y)\in\mathcal{V}} X = \mathcal{X}$ (otherwise we may replace $\mathcal{X}$ by the sum).

1° Let $x \in \mathcal{X}$ be arbitrary. From [Rud74, Theorem 12.6] we conclude that $\|x\|^2_{\mathcal{X}} = \sum_{(X,Y)\in\mathcal{V}} \|P_X x\|^2_X$, and that $P_X x = 0$ for all but countably many $(X,Y) \in \mathcal{V}$. Since $\|P_Y^* F_{X,Y} P_X x\| \le M\|P_X x\|$, the sum $\sum_{(X,Y)\in\mathcal{V}} P_Y^* F_{X,Y} P_X x$ converges (so F is well defined) and

$$\|Fx\|^2_{\mathcal{Y}} = \sum_{(X,Y)\in\mathcal{V}} \|P_Y F x\|^2_Y \le \sum_{(X,Y)\in\mathcal{V}} M^2 \|P_X x\|^2_X = M^2\|x\|^2_{\mathcal{X}}. \tag{11}$$

Clearly F is also linear. Thus, $F \in \mathcal{B}$ and $\|F\|_{\mathcal{B}} \le M$; obviously, also $\|F\|_{\mathcal{B}} \ge M$.

Equation (8) is obvious. It follows that

$$\langle y, Fx\rangle_{\mathcal{Y}} = \sum_{\mathcal{V}} \langle y, P_Y^* F_{X,Y} P_X x\rangle = \sum_{\mathcal{V}} \langle y, \tilde{P}_Y^* F \tilde{P}_X x\rangle, \tag{12}$$

so also the latter equation in (9) holds. Consequently,

$$\langle F^* y, x\rangle_{\mathcal{X}} = \sum_{\mathcal{V}} \langle P_X^* F^* P_Y y, x\rangle_{\mathcal{X}}, \tag{13}$$

hence (9) holds.

2° *Case $\mathcal{H}^\infty$ in place of $\mathcal{B}$:* (Case $\mathcal{H}^\infty_-$ obviously follows.) From 1° it follows that now F is a function $\mathbb{D} \to \mathcal{B}(\mathcal{X},\mathcal{Y})$, bounded by M. If $(X,Y) \in \mathcal{V}$ and $x \in X$, then $\langle y, Fx\rangle_{\mathcal{Y}} = \langle P_Y y, F_{X,Y} x\rangle_Y$ is holomorphic for each $y \in \mathcal{Y}$, hence F is holomorphic [HP57, Theorem 3.10.1] (because the span of such x is dense in $\mathcal{X}$). Equations (8) and (9) obviously follow from 1° (applied to $F(z)$ for each $z \in \mathbb{D}$).

3° *Case $\mathrm{L}^\infty_{\mathrm{strong}}$ in place of $\mathcal{B}$:* Choose first the representatives $F_{X,Y}$ so that $\|F_{X,Y}\|_{\mathcal{B}(X,Y)} \le M$ for each $(X,Y) \in \mathcal{V}$ (Proposition 2.2). By 1°, F becomes a function $\mathbb{T} \to \mathcal{B}(\mathcal{X},\mathcal{Y})$ with $\sup_{t\in\mathbb{T}} \|F(t)\| \le M$. Being a countable sum (bounded by $M\|x\|$) of (orthogonal-range) L^∞ functions, Fx is measurable, for each $x \in \mathcal{X}$; linearity is obvious. Obviously, $\|F\|_{\mathrm{L}^\infty_{\text{strong}}} \geq M$, hence $\sup_{t\in\mathbb{T}} \|F(t)\| = M = \|F\|_{\mathrm{L}^\infty_{\text{strong}}}$. Now (8) and (9) are obvious pointwise, hence (8) also holds in $\mathrm{L}^\infty_{\text{strong}}$ and, quite obviously, $M_{F^*} = M_F^*$, i.e.,

$$\langle Fu, v\rangle_{\mathrm{L}^2} = \langle u, F^* v\rangle_{\mathrm{L}^2} \qquad \text{for all } u \in \mathrm{L}^2(\mathcal{X}),\ v \in \mathrm{L}^2(\mathcal{Y}). \tag{14}$$

Let then $F'_{X,Y}$ be an arbitrary element of the class of $F_{X,Y}$, for each $(X,Y) \in \mathcal{V}$. Fix some $x \in \mathcal{X}$. Then $F'_{X,Y} P_X x = F_{X,Y} P_X x$ a.e. for each $(X,Y) \in \mathcal{V}$, and $P_X x \neq 0$ for at most countably many X, hence the sum $\sum P_Y^* F_{X,Y} P_X x$ indeed converges in $\mathrm{L}^\infty(\mathcal{Y})$, independently of the representatives.

(a1) We prove the case $\mathrm{L}^\infty_{\text{strong}}$ below. To obtain the proof for $\mathcal{H}^\infty$, $\mathcal{H}^\infty_-$ or $\mathcal{H}^2_{\text{strong}}$, replace $\mathbb{T}$ by $\mathbb{D}$. For $\mathcal{B}$ the proof is analogous.

For the case $g \in \mathcal{X}$, the claims were established in 1° (except the third one, which follows by definition). Assume then $g \in \mathrm{L}^2(\mathcal{X})$. Since (use Proposition 2.4)

$$\|g\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{T}} \|g\|_{\mathcal{X}}^2\, dm = \frac{1}{2\pi}\int_{\mathbb{T}} \sum_{\mathcal{V}} \|P_X g\|_{\mathcal{X}}^2\, dm = \sum_{\mathcal{V}} \|P_X g\|_2^2, \tag{15}$$

we have $P_X g = 0$ a.e. for all but countably many $(X,Y) \in \mathcal{V}$. The remaining claims obviously follow by applying the case $g \in \mathcal{X}$ to each $g(z)$, $z \in \mathbb{T}$. (The last claim is obvious if the sum is finite, and finite subsums converge in $\mathrm{L}^2(\mathcal{X})$.) In the L^2 case the sums converge both pointwise (a.e.) and in L^2 (recall [Rud74, Theorem 12.6]).

(a2) The first claim was given in (7). If $x \in \mathcal{X}$, then

$$GFx = G\sum P_Y^* F_{X,Y} P_X x = \sum P_{Z_Y}^* G_{Y,Z_Y} F_{X,Y} P_X x. \tag{16}$$

(a3) Let $\{x_1, x_2, \ldots\} \subset \tilde{X}$ be dense. The set $\tilde{\mathcal{V}}^{\mathcal{X}} := \{X \mid P_X x_k \neq 0$ for some $(X,Y) \in \mathcal{V}$ and some $k\}$ is countable, by (a1), and $\{x_1, x_2, \ldots\} \subset \sum_{X\in\mathcal{V}^{\mathcal{X}}} X$, hence $\tilde{X} \subset \sum_{X\in\mathcal{V}^{\mathcal{X}}} X$.

(b2) Except for (7), which we establish below, the proof is analogous to 3° and hence omitted. For each $x \in \mathcal{X}$, we have $\|G(z)x\|_{\mathcal{Y}}^2 = \sum \|G_{X,Y}(z) P_X x\|_Y^2$ a.e. (see the proof of (a1)), hence

$$\|Gx\|_2^2 = \sum \|G_{X,Y} P_X x\|_2^2 \leq \sum M \|P_X x\|_2^2 = M\|x\|_{\mathcal{X}}^2. \tag{17}$$

□

By $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ we denote the set of holomorphic functions $\mathbb{D} \to \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Now we establish the diagonalization (1), i.e., the converse to Theorem 3.1:

Theorem 3.2 ($F \mapsto \{F_{X,Y}\}$)

(a) *Let $F \in \mathcal{H}(\mathcal{X}, \mathcal{Y})$. Then there exists a collection $\mathcal{V}$ of pairs (X, Y), where the spaces X (resp., Y) are pairwise orthogonal closed separable subspaces of $\mathcal{X}$ (resp., $\mathcal{Y}$) such that $\tilde{P}_Y F \tilde{P}_X = \tilde{P}_Y F = F \tilde{P}_X$ for each $(X, Y) \in \mathcal{V}$, and $\mathcal{X} = \sum_{(X,Y)\in\mathcal{V}} X$, $\mathcal{Y} = \sum_{(X,Y)\in\mathcal{V}} Y$. If $\mathcal{X} = \mathcal{Y}$, then we can, in addition, have $X = Y$ for every $(X, Y) \in \mathcal{V}$.*

(b) *If F and $\mathcal{V}$ are as above and we set $F_{X,Y} := P_Y F P_X^*$ for each $(X, Y) \in \mathcal{V}$, then $F = \sum_{(X,Y)\in\mathcal{V}} P_Y^* F_{X,Y} P_X$.*

(c) *Parts (a) and (b) also hold with $\mathcal{B}$, $\mathrm{L}^\infty_{\mathrm{strong}}$, $\mathcal{H}^\infty$, $\mathcal{H}^\infty_-$ or $\mathcal{H}^2_{\mathrm{strong}}$ in place of $\mathcal{H}$; note that then Theorem 3.1 applies.*

(d) *Parts (a) and (b) also hold with $\mathrm{L}^1_{\mathrm{strong}}$ in place of $\mathcal{H}$.*

(e) *Parts (a) and (b) also hold if, instead of $F \in \mathcal{H}(\mathcal{X}, \mathcal{Y})$, we assume that $F \in \mathcal{B}(A(\mathcal{X}), B(\mathcal{Y}))$, where A and B each stand for one of L^2, $\mathcal{H}^2$ or $\mathcal{H}^2_-$. Moreover, then $f = \sum \tilde{P}_X f$ and $g = \sum \tilde{P}_Y g$ for all $f \in A(\mathcal{X})$, $g \in B(\mathcal{Y})$, and $\mathrm{Ran}(F) = \sum P_Y^* \,\mathrm{Ran}(F_{X,Y})$, hence $\mathrm{rank}(F) = \sum \mathrm{rank}(F_{X,Y})$.*

(f) *Let $\mathcal{J}$ (resp., $\mathcal{K}$) denote the set of closed subspaces of $\mathcal{X}$ (resp., $\mathcal{Y}$). Part (e) holds even if, instead of A and B being L^2, $\mathcal{H}^2$ or $\mathcal{H}^2_-$, we assume that A and B are as in Lemma B.1 and require*
a. that $A(X)$ and $B(Y)$ are Hilbert spaces for all $X \in \mathcal{J}$, $Y \in \mathcal{K}$,
b. that for any $X, \tilde{X} \in \mathcal{J}$ such that $\tilde{X} \subset X$ we have $P_{\tilde{X}} A(X) = A(\tilde{X})$ and $A(X) = A(\tilde{X}) \oplus A(\tilde{X}_2)$, where $\tilde{X}_2 := \{x \in X \mid x \perp \tilde{X}\}$, and
c. that the same requirements are be satisfied by $\mathcal{K}$ (resp., B) in place of $\mathcal{J}$ (resp., A).

(g) *Given any separable subsets $X_0 \subset \mathcal{X}$ and $Y_0 \subset \mathcal{Y}$, we may choose $\mathcal{V}$ so that, in addition to the requirements of any one of (a)–(f), we have $X_0 \subset X$ and $Y_0 \subset Y$ for some $(X, Y) \in \mathcal{V}$.*

(The proof is given in Appendix B. Recall that $X = X_1 \oplus X_2$ means that X_1 and X_2 are closed subspaces of the Hilbert space X, $X_1 \perp X_2$, and $X = X_1 + X_2 := \{x_1 + x_2 \mid x_1 \in X_1 \text{ and } x_2 \in X_2\}$.)

As before, the sums run over $\mathcal{V}$. Note that $\tilde{P}_Y F = F \tilde{P}_X$ is equivalent to $\tilde{P}_Y F \tilde{P}_X = \tilde{P}_Y F = F \tilde{P}_X$ (because then $\tilde{P}_Y F \tilde{P}_X = \tilde{P}_Y \tilde{P}_Y F = \tilde{P}_Y F$).

Note also that the identity $\tilde{P}_Y F \tilde{P}_X = \tilde{P}_Y F = F \tilde{P}_X$ is claimed to hold in the space considered, not pointwise everywhere for each representative (in the case of $\mathrm{L}^1_{\mathrm{strong}}$ or $\mathrm{L}^\infty_{\mathrm{strong}}$). Thus, in the case of $\mathrm{L}^\infty_{\mathrm{strong}}$, we just have $\|\tilde{P}_Y F - F \tilde{P}_X\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = 0$, i.e., $\tilde{P}_Y F x = F \tilde{P}_X x$ in $\mathrm{L}^\infty(\mathcal{Y})$ (that is, a.e. on $\mathbb{T}$) for each $x \in \mathcal{X}$. In particular, $F = \sum P_Y^* F_{X,Y} P_X$ holds in $\mathrm{L}^\infty_{\mathrm{strong}}$ but we need not have $F(z) = \sum P_Y^* F_{X,Y}(z) P_X$ for any $z \in \mathbb{T}$ (cf. Theorem 3.1(b1)).

Sometimes we have a sequence of functions instead of one F. That is not a problem: if we choose $\mathcal{V}_1$ for F_1, $\mathcal{V}_2$ for F_2 etc. as in Theorem 3.2, then we can "combine" them to a collection $\mathcal{V}$ that suits to each of the functions:

Lemma 3.3 **($\{\mathcal{V}_j\} \Rightarrow \mathcal{V}$)** *Let, for each $j \in \{1, 2, \ldots\}$, the set $\mathcal{V}_j$ be a collection of pairs (X, Y), where the spaces of X (resp., Y) are pairwise orthogonal closed separable subspaces of $\mathcal{X}$ (resp., $\mathcal{Y}$) that satisfy $\mathcal{X} = \sum_{(X,Y)\in\mathcal{V}_j} X$, $\mathcal{Y} = \sum_{(X,Y)\in\mathcal{V}_j} Y$.*

Then there is a collection $\mathcal{V}$ satisfying the same assumptions with the additional property that for each j and each $(X, Y) \in \mathcal{V}_j$, we have $X \subset X'$ and $Y \subset Y'$ for some $(X', Y') \in \mathcal{V}$.

It follows that if some $\mathcal{V}_j$ and some F relate as in Theorem 3.2, then so do $\mathcal{V}$ and F (i.e., $\tilde{P}_Y F = F\tilde{P}_X$ for each $(X, Y) \in \mathcal{V}$) provided that $F\sum_X \tilde{P}_X = \sum_X F\tilde{P}_X$ and $\sum_Y P_Y F = \sum_Y \tilde{P}_Y F$ for countable sums of separable orthogonal subspaces (this is satisfied in the settings (a)–(e) of Theorem 3.2).

Moreover, if $\mathcal{X} = \mathcal{Y}$ and $X = Y$ for each $(X, Y) \in \mathcal{V}_j$ and each j, then we can have $X = Y$ for each $(X, Y) \in \mathcal{V}$.

(The proof is given at the end of Appendix B.)

In Theorem 4.3 we shall see that $\|M_F\|_{\mathcal{B}} = \|F\|_{\mathrm{L}^\infty_{\text{strong}}}$. This motivates us to define the *adjoint* $F^* \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{Y}, \mathcal{X})$ of $F \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{X}, \mathcal{Y})$ through $M_{F^*} = M_F^*$ (as in (14), cf. Theorems 3.1(b1), 4.3 and 4.10). Further motivation will be given below. At this point we should add the condition "if such a function (class) F^* exists", but we shall soon see that condition being redundant.

If $[f] \in \mathrm{L}^\infty(\mathcal{B}(\mathcal{X}, \mathcal{Y}))$, then $[f^*] \in \mathrm{L}^\infty(\mathcal{B}(\mathcal{Y}, \mathcal{X}))$. However, a function f such that $[f] \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{X}, \mathcal{Y})$ may have f^*y non-Bochner-measurable for some $y \in \mathcal{Y}$, by Example C.2(b)&(c) (even if $[f] = [0]$ and $\operatorname{ess\,sup} \|f^*\|_{\mathcal{B}(\mathcal{Y},\mathcal{X})} = \infty$). Nevertheless, when f^* is strongly measurable (or equivalently, almost separably-valued), it gives us the adjoint of $[f]$:

Lemma 3.4 *Let $[f] \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{X}, \mathcal{Y})$. If f^*y is measurable for every $y \in \mathcal{Y}$, then $\|f^*\|_{\mathrm{L}^\infty_{\text{strong}}} = \|f\|_{\mathrm{L}^\infty_{\text{strong}}}$ and $[f^*] = [f]^*$.*

Proof: By Lemma A.9, we have

$$\begin{aligned}\|fx\|_{\mathrm{L}^\infty_{\text{strong}}} &= \sup_{\|x\|_{\mathcal{X}}\le 1,\ \|y\|_{\mathcal{Y}}\le 1} \|fx\|_\infty = \sup_{\|x\|_{\mathcal{X}}\le 1,\ \|y\|_{\mathcal{Y}}\le 1} \|\langle fx, y\rangle\|_\infty \\ &= \sup_{\|x\|_{\mathcal{X}}\le 1,\ \|y\|_{\mathcal{Y}}\le 1} \|\langle x, f^*y\rangle\|_\infty = \sup_{x\in\mathcal{X},\ y\in\mathcal{Y}} \|f^*y\|_\infty = \|f^*\|_{\mathrm{L}^\infty_{\text{strong}}},\end{aligned}$$

so $f^* \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{Y}, \mathcal{X})$.

It is obvious from (14) that $M_{[f^*]} = M_{[f]}^*$. □

To get such a nice representative f, we can choose it in the following canonical ("diagonalized") way:

Corollary 3.5 *Let $F \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$. Choose $\mathcal{V}$ and the classes $F_{X,Y}$ as in Theorem 3.2(c). For each class $F_{X,Y}$, choose a representative $f_{X,Y}$ that satisfies $\sup_{\mathbb{T}} \|f_{X,Y}\|_{\mathcal{B}(X,Y)} = \|F_{X,Y}\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$. Then a representative f of F is given by $f(z) := \sum_{(X,Y)\in\mathcal{V}} P_Y^* f_{X,Y}(z) P_X$ $(z \in \mathbb{T})$. Moreover,*

$$f^* = \sum_{(X,Y)\in\mathcal{V}} P_X^* f_{X,Y}^* P_Y \tag{18}$$

(also here the sum can be computed pointwise and it lies in $\mathcal{B}(\mathcal{Y},\mathcal{X})$ for every $z \in \mathbb{T}$).

Furthermore, $[f^*] = [f]^* \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{Y},\mathcal{X})$, $\langle x, f^*y\rangle = \langle Fx, y\rangle$ a.e. for each $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, and $\langle u, f^*v\rangle_{\mathrm{L}^2} = \langle Fu, v\rangle_{\mathrm{L}^2}$ for each $u \in \mathrm{L}^2(\mathcal{X})$ and $v \in \mathrm{L}^2(\mathcal{Y})$ (i.e., $M_{f^*} = M_f^*$).

Thus, here the pointwise sum (resp., adjoint) equals the sum (resp., adjoint) in $\mathrm{L}^\infty_{\mathrm{strong}}$.
Proof: By Theorem 3.1, $f(z) \in \mathcal{B}(\mathcal{X},\mathcal{Y})$ and $\|f(z)\| \le \|F\|$ $\forall z \in \mathbb{T}$. But $P_Y^* f_{X,Y} P_X x = P_Y^* F_{X,Y} P_X x$ a.e. (and these are zero for all but countably many $(X,Y) \in \mathcal{V}$), so fx is strongly measurable and $[f] = F \in \mathrm{L}^\infty_{\mathrm{strong}}$. We obtain $f^* \in \mathrm{L}^\infty_{\mathrm{strong}}$ analogously and the equality in (18) pointwise from (9) The rest holds because $fx = Fx$ a.e. and $fu = Fu$ a.e. (also $Fu \in \mathrm{L}^\infty$ is independent of the representative of F, because u is almost separably-valued). $\square$

(The condition $\sup_{\mathbb{T}} \|f_{X,Y}\|_{\mathcal{B}(X,Y)} = \|F_{X,Y}\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$ in Corollary 3.5 is not extraneous, by C.4.)

In the separable case any representative of F will give us the adjoint:

Lemma 3.6 *Let $\mathcal{X}$ be separable and $[f],[g] \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$. Then $[f]=[g]$ iff $f = g$ a.e. Thus, then $[f^*] = [h^*]$ for every $h \in [f]$.*

Proof: If $S \subset \mathcal{X}$ is dense and countable and $fx = gx$ on $\mathbb{T} \setminus N_x$, where $m(N_x) = 0$, for each $x \in \mathcal{X}$, then $f = g$ on $\mathbb{T} \setminus \cup_{x\in S} N_x$. The converse is obvious. $\square$

Compositions of operators are well defined: if $[G] \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{Y},\mathcal{Z})$, then $[G][F] := [GF] \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Z})$ with $\|[GF]\| \le \|[G]\|\|[F]\|$ [Mik08, Corollary 2.3]. The class of GF is thus independent of the representatives F and G). In particular, $\mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{X})$ is a B^*-algebra.

By $\mathcal{C}(\mathcal{X},\mathcal{Y})$ we denote the set of continuous functions $\mathbb{T} \to \mathcal{B}(\mathcal{X},\mathcal{Y})$.

Lemma 3.7 *Let $F \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ and choose $\mathcal{V}$ as in Theorem 3.2. Then $F \in \mathcal{C}(\mathcal{X},\mathcal{Y})$ iff the functions $F_{X,Y}$ are equicontinuous.*

A third equivalent condition is that the functions are uniformly equicontinuous.
Proof: If $\|F(t) - F(t')\| < \epsilon$, then

$$\|F_{X,Y}(t) - F_{X,Y}(t')\| = \|P_Y(F(t) - F(t'))P_X^*\| < \epsilon \tag{19}$$

for each $(X, Y) \in \mathcal{V}$. Conversely, if $\epsilon > 0$ is given and $|t - t'| < \delta \implies \|F_{X,Y}(t) - F_{X,Y}(t')\| < \epsilon$ for each $(X, Y) \in \mathcal{V}$, then (by Theorem 3.1(a1))

$$\|F(t)x - F(t')x\|^2 = \sum_{\mathcal{V}} \|F_{X,Y}(t)x - F_{X,Y}(t')x\|^2 \leq \sum_{\mathcal{V}} \epsilon^2 \|P_X x\|^2 = \epsilon^2 \|x\|^2 \tag{20}$$

when $|t - t'| < \delta$ and $x \in \mathcal{X}$, hence then F is continuous. □

4 Results for the unit circle

In this section we extend to the nonseparable case mostly standard results on the Hankel and Toeplitz operators of operator-valued functions in $\mathcal{H}^\infty$, $\mathrm{L}^\infty_{\mathrm{strong}}$ or $\mathcal{H}^2_{\mathrm{strong}}$. We work on the unit disc (or circle); corresponding results for the half-plane (or real line) are given in Section 5.

As above, $\mathcal{X}$ and $\mathcal{Y}$ denote arbitrary (possibly nonseparable) Hilbert spaces. It is well known [Mik02, p. 977] that square-integrable functions on the unit circle $\mathbb{T}$ are exactly those with ℓ^2 Laurent series coefficients:

$$\mathrm{L}^2(\mathcal{X}) = \{f = \sum_{k=-\infty}^{\infty} z^k x_k \mid \|f\|_2^2 := \sum_k \|x_k\|^2_{\mathcal{X}} < \infty\}. \tag{21}$$

Moreover, $\mathcal{H}^2$ (resp., $\mathcal{H}^2_-$) consists of those series where $x_k = 0$ for all $k < 0$ (resp., $k \geq 0$), by Propositions 2.3 and A.6.

Definition 4.1 **(**$P_+,\ P_-,\ S$**)** *By* $P_+ : \sum_{k=-\infty}^{\infty} z^k x_k \mapsto \sum_{k=0}^{\infty} z^k x_k$ *we denote the orthogonal projection* $\mathrm{L}^2 \to \mathcal{H}^2$*, and we set* $P_- := I - P_+$*. When* f *is a function on* $\mathbb{D}$ *or on* $\mathbb{T}$*, we set*

$$(Sf)(z) := zf(z), \qquad (S^*f)(z) := z^{-1}f(z). \tag{22}$$

Definition 4.2 *The operators* $\Gamma \in \mathcal{B}(\mathcal{H}^2_-(\mathcal{X}), \mathcal{H}^2(\mathcal{Y}))$ *that satisfy* $P_+S^*\Gamma = \Gamma S^*$ *are called* Hankel operators.

The Hankel operator $\Gamma_F \in \mathcal{B}(\mathcal{H}^2_-(\mathcal{X}), \mathcal{H}^2(\mathcal{Y}))$ of *an essentially bounded measurable function* $F \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X}, \mathcal{Y})$ *is defined by*

$$\Gamma_F := P_+ M_F P_-. \tag{23}$$

Such an operator is a Hankel operator:

$$\Gamma_F S^* P_- = P_+ F P_- S^* P_- = P_+ F S^* P_- = P_+ S^* F P_- = P_+ S^* P_+ F P_- = P_+ S^* \Gamma_F. \tag{24}$$

In the literature [FF90] [Pel03], the map $\Gamma' := \Gamma\mathcal{R} \in \mathcal{B}(\mathcal{H}^2(\mathcal{X}), \mathcal{H}^2(\mathcal{Y}))$ is often used in place of Γ, where the linear isometry $\mathcal{R} \in \mathcal{B}(\mathcal{H}^2, \mathcal{H}^2_-)$ is defined by $\mathcal{R} : \sum_{k=0}^{\infty} z^k x_k \mapsto \sum_{k=-\infty}^{-1} z^k x_{-1-k}$. Then the Hankel condition becomes $P_+S^*\Gamma' = \Gamma' S$.

Lemma 13.1.5 of [Mik02] says that bounded, linear, shift-invariant operators $\mathrm{L}^2(\mathcal{X}) \to \mathrm{L}^2(\mathcal{Y})$ are exactly the multiplication operators induced by the $\mathrm{L}^\infty_{\mathrm{strong}}$ functions $\mathbb{T} \to \mathcal{B}(\mathcal{X}, \mathcal{Y})$:

Theorem 4.3 (SI$=\mathrm{L}^\infty_{\text{strong}}$) *The operators $T \in \mathcal{B}(\mathrm{L}^2(\mathcal{X}), \mathrm{L}^2(\mathcal{Y}))$ that satisfy $ST = TS$ are exactly the operators of the form $M_F : f \mapsto Ff$, where $F \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{X}, \mathcal{Y})$. Moreover, $\|M_F\|_{\mathcal{B}} = \|F\|_{\mathrm{L}^\infty_{\text{strong}}}$ for every $F \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{X}, \mathcal{Y})$.*

(This could also be deduced from Theorem 1.2 of [Mik08] or proved analogously.)

Now we can generalize standard results on Hankel operators, including the Nehari or Page Theorem:

Theorem 4.4 ($\Gamma = \Gamma_F$) *An operator $\Gamma : \mathcal{H}^2_-(\mathcal{X}) \to \mathcal{H}^2(\mathcal{Y})$ is a Hankel operator iff $\Gamma = \Gamma_F$ for some $F \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{X}, \mathcal{Y})$. If Γ is a Hankel operator, then we can choose F so that $\Gamma = \Gamma_F$ and $\|F\|_{\mathrm{L}^\infty_{\text{strong}}} = \|\Gamma\|_{\mathcal{B}}$. Finally, if $F, \tilde{F} \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{X}, \mathcal{Y})$, then $\Gamma_F = \Gamma_{\tilde{F}}$ iff $F - \tilde{F} \in \mathcal{H}^\infty_-(\mathcal{X}, \mathcal{Y})$.*

The claim $F - \tilde{F} \in \mathcal{H}^\infty_-$ means that the class $F - \tilde{F}$ contains an element of $\mathcal{H}^\infty_-$, i.e., that there exists $G \in \mathcal{H}^\infty_-$ such that $(F - \tilde{F})x = Gx$ a.e. on $\mathbb{T}$ for each $x \in \mathcal{X}$.

Obviously, always $\|F\|_{\mathrm{L}^\infty_{\text{strong}}} \geq \|\Gamma_F\|$. Note also that we have defined $\mathcal{H}^\infty$, $\mathcal{H}^\infty_-$ and $\mathcal{H}^2$ so that they contain the constant functions ($F \equiv A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$) whereas $\mathcal{H}^2_-$ does not.

Proof of Theorem 4.4: 1° Theorem 4.4 is well known in the case of separable $\mathcal{X}$ and $\mathcal{Y}$ (see, e.g., Theorem 3.3 and Lemma 3.2 of [FF90]), hence we can and will refer to its separable case in this proof. Since "if" was noted above, we assume that Γ is a Hankel operator.

2° Choose $\mathcal{V}$ for Γ as in Theorem 3.2(e). For each $(X, Y) \in \mathcal{V}$, the operator

$$\Gamma_{X,Y} := P_Y \Gamma P_X^* \in \mathcal{B}(\mathcal{H}^2_-(X), \mathcal{H}^2(Y)) \tag{25}$$

is a Hankel operator (because P_X^* and P_Y commute with P_+, P_- and S), By Theorem 4.4 (which can be applied, by separability and 1°), it follows that there exists $F_{X,Y} \in \mathrm{L}^\infty(X, Y)$ such that $\Gamma_{X,Y} := P_Y \Gamma P_X$ equals $\Gamma_{F_{X,Y}}$ and $\|F_{X,Y}\|_{\mathrm{L}^\infty_{\text{strong}}} = \|\Gamma_{X,Y}\| \leq \|\Gamma\|$. Set

$$F := \sum_{(X,Y) \in \mathcal{V}} P_Y^* F_{X,Y} P_X \tag{26}$$

to have, by Theorem 3.1, that $F \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{X}, \mathcal{Y})$, $\|F\| \leq \|\Gamma\|$ and

$$\Gamma_F \tilde{P}_X = \tilde{P}_Y \Gamma_F \tilde{P}_X = P_Y^* \Gamma_{F_{X,Y}} P_X = P_Y^* \Gamma_{X,Y} P_X = \tilde{P}_Y \Gamma \tilde{P}_X = \Gamma \tilde{P}_X \tag{27}$$

for each $(X, Y) \in \mathcal{V}$. Consequently, $\Gamma_F f = \Gamma f$ for each $f \in \mathcal{H}^2_-(\mathcal{X})$, i.e., $\Gamma_F = \Gamma$. Obviously, $\|\Gamma\| = \|P_+ F P_-\| \leq \|F\|_{\mathrm{L}^\infty_{\text{strong}}}$, hence $\|\Gamma\| = \|F\|_{\mathrm{L}^\infty_{\text{strong}}}$.

3° If $F \in \mathrm{L}^\infty_{\text{strong}}$ is such that $\Gamma_F = 0$ and we choose $\mathcal{V}$ for F as in Theorem 3.2(c), then $P_Y F P_X^* \in \mathcal{H}^\infty_-(X, Y)$ and $\|P_Y F P_X^*\| \leq \|F\|$ for each $(X, Y) \in \mathcal{X}$, by Theorem 4.4, hence $F \in \mathcal{H}^\infty_-(\mathcal{X}, \mathcal{Y})$, by Theorem 3.1(b1). $\square$

The one-block, optimal *Nehari Theorem* is as follows:

Corollary 4.5 (Nehari–Page) *For any $F \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ we have*

$$\|\Gamma_F\|_{\mathcal{B}(\mathcal{H}^2_-,\mathcal{H}^2)} = \min_{G\in\mathcal{H}^\infty_-} \|F-G\|_{\mathrm{L}^\infty_{\mathrm{strong}}}. \tag{28}$$

From Lemma A.8 we observe that the above contains a direct generalization of the Nehari (Page) Theorem for separable Hilbert spaces.

Proof: Necessarily $\|\Gamma_F\| = \|P_+(F-G)P_-\| \le \|F-G\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$ for any $G \in \mathcal{H}^\infty_-$, so $\|\Gamma_F\| \le \inf_{G\in\mathcal{H}^\infty_-}\|F-G\|$. By Theorem 4.4, there exists $\tilde F \in \mathrm{L}^\infty_{\mathrm{strong}}$ such that $\|\tilde F\| = \|\Gamma_F\|$ and $\Gamma_{\tilde F} = \Gamma_F$, hence $G := F - \tilde F \in \mathcal{H}^\infty_-$ and $\|F-G\| = \|\tilde F\| = \|\Gamma_F\|$. □

An infinite "matrix" A of the form $A_{j,k} = A_{j+k-1}$ $(j,k\ge 1)$ for some $\{A_k\}_{k=1}^\infty \subset \mathcal{B}(\mathcal{X},\mathcal{Y})$ is called a *Hankel matrix*. The operator $\Gamma : \sum_{k=-\infty}^{-1} z^k x_k \mapsto \sum_{k=0}^\infty z^k y_k$, where

$$y_k := \sum_{j=-\infty}^{-1} A_{k-j}x_j = \sum_{j=1}^{\infty} A_{j+k}x_{-j} = \sum_{j=1}^{\infty} A_{j,k+1}x_{-j}, \quad (k\ge 0), \tag{29}$$

(or "$y_{\cdot-1} = Ax_-$.") is (well-defined and) a Hankel operator iff there exists $F \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ such that $A_k = \widehat{F}(k)$ $(k\ge 1)$. Conversely, if $F \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$, then $A_k := \widehat{F}(k)$ (see Lemma A.5) determines a Hankel matrix that satisfies $\Gamma = \Gamma_F$. (This follows from Theorem 4.4 as in the separable case.) Functions $F, G \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ determine the same Hankel operator iff $\widehat{F}(k) = \widehat{G}(k)$ for all $k \ge 1$.

The *nth singular value* of $T \in \mathcal{B}(\mathcal{X},\mathcal{Y})$ is defined as

$$s_n(T) := \inf\{\|T-K\| \bigm| K \in \mathcal{B}(\mathcal{X},\mathcal{Y}),\ \mathrm{rank}(K) \le n-1\}. \tag{30}$$

Note that $\|T\| = s_1(T) \ge s_2(T) \ge \cdots$. The *Adamjan–Arov–Krein Theorem* says that if T is a Hankel operator, then inf = min and we can choose a minimizing K so that also it is a Hankel operator:

Theorem 4.6 (Adamjan–Arov–Krein) *For any $F \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ and $n \in \{1,2,\dots\}$ we have*

$$s_n(\Gamma_F) = \min_{G\in\mathrm{L}^\infty_{\mathrm{strong}}} \{\|F-G\|_{\mathrm{L}^\infty_{\mathrm{strong}}} \bigm| \mathrm{rank}\,\Gamma_G \le n-1\}. \tag{31}$$

For $n = 1$ this is exactly the Nehari Theorem, by Theorem 4.4. The problem of finding G satisfying (31) (or close enough) is called the *Nehari–Takagi problem* (or the Hankel norm approximation problem).

Proof: 1° Set $R := s_n(\Gamma_F)$. By the definition of $s_n(\Gamma_F)$, for each $k \in \mathbb{Z}_+ := \{1,2,\dots\}$ there exists $T^k \in \mathcal{B}(\mathcal{H}^2_-(\mathcal{X}),\mathcal{H}^2(\mathcal{Y}))$ such that $\|\Gamma_F - T^k\| < 1/k + R$ and $\mathrm{rank}\,T^k \le n-1$,

Choose $\mathcal{V}_0$ for Γ_F and $\mathcal{V}_1,\mathcal{V}_2,\dots$ for $T^1,T^2,\dots$ as in Theorem 3.2(c)&(e), and then a $\mathcal{V}$ to replace $\mathcal{V}_0,\mathcal{V}_1,\dots$ as in Lemma 3.3. For each $(X,Y)\in\mathcal{V}$ we set $\Gamma_{X,Y} := \Gamma_{F_{X,Y}} = P_+F_{X,Y}P_-$ and

$$g(X) := \min\{j \ge 1 \bigm| s_j(\Gamma_{X,Y}) \le R\} - 1. \tag{32}$$

Thus, $g(X)$ is the minimal Hankel dimension needed to make $\|\Gamma_{X,Y}\|$ smaller than R. We shall show in 2° that the sum of $g(X)$'s is at most $n-1$ (in particular, at most $n-1$ of them are nonzero), so that (in 3°) F can be fixed by fixing just those $\leq n-1$ components $F_{X,Y}$.

2° *We show that* $\sum_{(X,Y)\in\mathcal{V}} g(X) \leq n-1$*:*

To get a contradiction, assume that $\sum_{j=1}^m g(X_j) \geq n$ for some $m \in \mathbb{Z}_+$ and some pairs $(X_1, Y_1), (X_2, Y_2), \ldots, (X_m, Y_m) \in \mathcal{V}$. We have $s_{g(X)}(\Gamma_{X,Y}) > R$ for each $(X,Y) \in \mathcal{V}$ (we set $s_0(T) := \infty$ for any operator T), hence there exists $k \in \mathbb{Z}_+$ such that

$$1/k < \epsilon := \min_{j=1,2,\ldots,m} s_{g(X_j)}(\Gamma_{X_j,Y_j}) - R. \tag{33}$$

Set $T^k_{X_j,Y_j} := P_{Y_j} T^k P^*_{X_j}$ for each j. By Theorem 3.2(e), we have

$$\sum_{j=1}^m \operatorname{rank}(T^k_{X_j,Y_j}) \leq \operatorname{rank}(T^k) \leq n-1, \tag{34}$$

hence $\operatorname{rank}(T^k_{X_j,Y_j}) < g(X_j)$ for some j. By the definition of s_n, this implies that

$$\begin{aligned} s_{g(X_j)}(\Gamma_{X_j,Y_j}) &\leq \|\Gamma_{X_j,Y_j} - \Gamma_{T^k_{X_j,Y_j}}\| \leq \|\Gamma_F - \Gamma_{T^k}\| \\ &< R + 1/k < R + \epsilon \leq s_{g(X_j)}(\Gamma_{X_j,Y_j}), \end{aligned} \tag{35}$$

a contradiction, as required.

3° For each $(X,Y) \in \mathcal{V}$, choose $G_{X,Y} \in \mathrm{L}^\infty_{\text{strong}}(X,Y)$ so that $\|F_{X,Y} - G_{X,Y}\| \leq R$ and $\operatorname{rank}(\Gamma_{G_{X,Y}}) \leq g(X)$ (i.e., use the separable case of Theorem 4.6, which equals Theorem I of [Tre85], p. 57, given in English as Theorem 4.3.1 of [Pel03]). Since $\|G_{X,Y}\| \leq R + \|F\|$ for each (X,Y), we get $G \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{X},\mathcal{Y})$ and $\|F - G\| \leq R$ from Theorem 3.1(b1). Since $\operatorname{Ran}(\Gamma_G) = \sum P^*_Y \operatorname{Ran}(\Gamma_{G_{X,Y}})$, we have $\operatorname{rank}(\Gamma_G) = \sum \operatorname{rank}(\Gamma_{G_{X,Y}}) \leq \sum g(X) \leq n-1$, by 2°. □

A Hankel operator Γ_F is finite-dimensional iff the McMillan degree of $F \in \mathrm{L}^\infty_{\text{strong}}$ is finite. If F is a matrix-valued $\mathcal{H}^\infty$ function, then the degree is finite iff F is rational. If $\mathcal{X}$ or $\mathcal{Y}$ is infinite-dimensional, then the definition of the McMillan degree becomes rather technical, see [Pel03, p. 81] for details (due to [Tre85]). (Treil and Peller assume that $\mathcal{X}$ and $\mathcal{Y}$ are separable, but this does not reduce generality, by the last claim of Theorem 4.7 below.)

By $\mathcal{C}(\mathcal{BC}(\mathcal{X},\mathcal{Y}))$ we denote the set of continuous functions $\mathbb{T} \to \mathcal{BC}(\mathcal{X},\mathcal{Y})$, where $\mathcal{BC}$ stands for compact linear operators $\mathcal{X} \to \mathcal{Y}$.

Theorem 4.7 (Hartman) *Let $F \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$. Then the following are equivalent:*

- **(i)** Γ_F *is compact* $\mathcal{H}^2_-(\mathcal{X}) \to \mathcal{H}^2(\mathcal{Y})$*;*
- **(ii)** $F \in \mathcal{H}^\infty_-(\mathcal{X},\mathcal{Y}) + \mathcal{C}(\mathcal{BC}(\mathcal{X},\mathcal{Y}))$*;*
- **(iii)** $\Gamma_F = \Gamma_G$ *for some* $G \in \mathcal{C}(\mathcal{BC}(\mathcal{X},\mathcal{Y}))$*;*
- **(iv)** $\widehat{F}(n) = \widehat{G}(n)$ $(n \geq 1)$ *for some* $G \in \mathcal{C}(\mathcal{BC}(\mathcal{X},\mathcal{Y}))$*.*

If (iii) holds and $\epsilon > 0$, then we can choose the G in (iii) and (iv) so that $\|G\|_\infty < \|\Gamma_F\| + \epsilon$ and $G = \tilde{P}_Y G \tilde{P}_X$ for some closed separable subspaces $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$.

Proof: One easily obtains the implications (iv)$\Leftrightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i) as on p. 74 of [Pel03]. Thus, we may assume (i) and we only need to construct G, X and Y.

Since Γ_F is the uniform limit of finite-dimensional operators, we have $\Gamma_F = \tilde{P}_B \Gamma_F \tilde{P}_A$ for some closed separable subspaces $A \subset \mathcal{H}^2(\mathcal{X})$, $B \subset \mathcal{H}^2(\mathcal{Y})$. But $A \subset \mathcal{H}^2(X)$ and $B \subset \mathcal{H}^2(Y)$ for some closed, separable subspaces $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$ (take a dense countable subset of A (resp., B), consisting of separably-valued functions, and take the closed span of the union of their ranges). Thus, $\Gamma_F = \tilde{P}_Y \Gamma_F \tilde{P}_X$.

Set $\tilde{F} := P_Y F P_X^* \in \mathrm{L}^\infty_{\mathrm{strong}}(X,Y)$. Then $\Gamma_{\tilde{F}} = P_Y \Gamma_F P_X^*$, hence $\Gamma_{\tilde{F}}$ is compact, hence $\Gamma_{\tilde{F}} = \Gamma_{\tilde{G}}$ for some $\tilde{G} \in \mathcal{C}(\mathcal{BC}(X,Y))$ with $\|G\|_\infty < \|\Gamma_{\tilde{F}}\| + \epsilon = \|\Gamma_F\| + \epsilon$, by Theorem 10 of [Pag70] (or Theorem 2.4.1 and p. 75 of [Pel03]). Therefore, $G := P_Y^* \tilde{G} P_X$ has the required properties. $\square$

The Corona Theorem says that if(f) $F \in \mathcal{H}^\infty(\mathcal{X},\mathcal{Y})$ and $F(z)^*F(z) \geq \epsilon I$ for all $z \in \mathbb{D}$, then F is left-invertible in $\mathcal{H}^\infty$. Unfortunately, the "if" part holds only when $\mathcal{X}$ is finite-dimensional [Tre89] (or trivially when $\dim \mathcal{Y} < \dim \mathcal{X}$). However, the coercivity of the anti-Toeplitz operator is always a sufficient and necessary condition for left-invertibility. Moreover, a related result, Tolokonnikov's Lemma, says that a left-invertible $\mathcal{H}^\infty$ function (as in (ii) below) can be complemented to an invertible one (as in (iii)):

Theorem 4.8 (Tolokonnikov) *Let $F \in \mathcal{H}^\infty(\mathcal{X},\mathcal{Y})$. Then the following are equivalent:*

- **(i)** *The anti-Toeplitz operator $P_- F P_-$ is* coercive*, i.e., there exists $\epsilon > 0$ such that for every $g \in \mathcal{H}^2_-(\mathcal{X})$ we have*

$$\|P_- F P_- g\|_2 \geq \epsilon \|g\|_2. \tag{36}$$

- **(i')** *The multiplication operator $M_{F^{\mathrm{d}}}$ by $F^{\mathrm{d}} := F(\bar{\cdot})^*$ maps $\mathcal{H}^2(\mathtt{Y})$ onto $\mathcal{H}^2(\mathtt{X})$.*
- **(ii)** $GF = I$ *for some* $G \in \mathcal{H}^\infty(\mathcal{Y},\mathcal{X})$*.*
- **(iii)** *There exist a closed subspace $\mathcal{Z} \subset \mathcal{Y}$ and a function $\tilde{F} \in \mathcal{H}^\infty(\mathcal{Z},\mathcal{X})$ such that $\begin{bmatrix} F & \tilde{F} \end{bmatrix} \in \mathcal{H}^\infty(\mathcal{X} \times \mathcal{Z},\mathcal{Y})$ is invertible in $\mathcal{H}^\infty$.*

Assume (i). Then the best possible norm of a left-inverse G in (ii) is $1/\epsilon$ for the maximal ϵ in (36). Set $M := \|F\|$. In (iii), we can have $\|\tilde{F}\| = 1$, $\| \begin{bmatrix} F & \tilde{F} \end{bmatrix} \| \le \sqrt{M^2+1}$, and (if $\mathcal{X} \neq \{0\}$)

$$\| \begin{bmatrix} F & \tilde{F} \end{bmatrix}^{-1} \|_{\mathcal{H}^\infty} \le \frac{M}{\epsilon}\sqrt{1+\epsilon^{-2}}. \tag{37}$$

By $GF = I$ in (ii) we mean that $G(z)F(z) = I$ for each $z \in \mathbb{D}$, or equivalently, that $GFx = x$ a.e. on $\mathbb{T}$ for each $x \in \mathcal{X}$ (Proposition 2.4).

Proof: Observe first that (i') is equivalent to (i).

1° For the separable case, the equivalence of (i)–(iii) and the fact that we can have $\|G\| = 1/\epsilon$ were established in Theorems 1.2 and 2.1 of [Tre04] (if we drop "$\subset \mathcal{Y}$"). We explain below the norm estimates for (iii) in the separable case.

By the proof of Lemma 6.1 of [Tre04], we have $\|\mathcal{P}\| \le M/\epsilon$. Therefore, $\|I - \mathcal{P}\| \le M/\epsilon$, by Lemma A.10 (if $\mathcal{X} \neq 0$). Since (in the middle of that proof) Θ is inner, we have $\|\Theta\| = 1$ and $\|R\| = \|\mathcal{Q}\| = \|I - \mathcal{P}\| \le M/\epsilon$ and $\|\tilde{F}\| \le 1$ (near the end of the proof), hence $\| \begin{bmatrix} F & \tilde{F} \end{bmatrix} \| \le \sqrt{M^2+1}$. As mentioned above, we can have $\|G\| = 1/\epsilon$, which leads to

$$\|\tilde{G}\| = \| \begin{bmatrix} G\mathcal{P} \\ R \end{bmatrix} \| \le \sqrt{(\epsilon^{-1} \cdot \epsilon^{-1}M)^2 + (\epsilon^{-1}M)^2} = \epsilon^{-1}M\sqrt{1+\epsilon^{-2}}. \tag{38}$$

2° Since implications (iii)⇒(ii)⇒(i) are obvious (take $\epsilon := 1/\|P_- G P_-\|$ and observe from Theorem 4.4 that $P_- G = P_- G P_-$), we assume (i). Apply Theorem 3.2(a)&(c) to $F \in \mathcal{H}^\infty$, and then find, for each $(X,Y) \in \mathcal{V}$, a Hilbert space Z and functions $\tilde{F}_{X,Y} \in \mathcal{H}^\infty(Z,Y)$, $K_{X,Y} \in \mathcal{H}^\infty(Y, X \times Z)$ such that

$$K_{X,Y} \begin{bmatrix} F_{X,Y} & \tilde{F}_{X,Y} \end{bmatrix} = I, \qquad \begin{bmatrix} F_{X,Y} & \tilde{F}_{X,Y} \end{bmatrix} K_{X,Y} = I, \tag{39}$$

$\|K_{X,Y}\| \le \epsilon^{-1}M\sqrt{1+\epsilon^{-2}}$, and $\|\tilde{F}_{X,Y}\| \le 1$. Let $\mathcal{Z} \subset \prod_{(X,Y)\in\mathcal{V}} Z$ be as in Lemma A.3, $\tilde{F} := \sum P_Y^* \tilde{F}_{X,Y} P_Z$ (Theorem 3.1(b1)), $K := \sum P_{X\times Z}^* K_{X,Y} P_Y$ to have

$$\|K\| \le \epsilon^{-1}M\sqrt{1+\epsilon^{-2}}, \quad \|\tilde{F}\| \le 1, \quad K \begin{bmatrix} F & \tilde{F} \end{bmatrix} = I, \quad \text{and} \quad \begin{bmatrix} F & \tilde{F} \end{bmatrix} K = I, \tag{40}$$

by Theorem 3.1(a2).

By Lemma A.2, $\mathcal{Z}$ is unitarily equivalent to a subspace, say $\tilde{\mathcal{Y}}$, of $\mathcal{Y}$. Let $T \in \mathcal{B}(\mathcal{Z}, \tilde{\mathcal{Y}})$ be such an isometry and replace $\tilde{F}$ by $\tilde{F}T$ and $\mathcal{Z}$ by $\tilde{\mathcal{Y}}$ to complete (iii).

3° *The estimate in (ii):* By Theorem 1.2 of [Tre04], we can have $\|G_{X,Y}\| \le 1/\epsilon$ for a left inverse $G_{X,Y} \in \mathcal{H}^\infty(Y,X)$ of $F_{X,Y}$ (see 2°), for each $(X,Y) \in \mathcal{V}$. Apply Theorem 3.1(b1)&(a2) to obtain $G \in \mathcal{H}^\infty(\mathcal{Y},\mathcal{X})$ such that $GF = I$ and $\|G\| \le 1/\epsilon$. Obviously, $\epsilon \ge 1/\|P_- G P_-\| \ge 1/\|G\|$, hence $\|G\| = 1/\epsilon$ is the minimal norm of a left inverse. □

If, in Theorem 4.8, we have $F^*F = I$ (cf. Theorem 4.10 below), then there are two more equivalent conditions:

Theorem 4.9 *Let $F \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{X},\mathcal{Y})$ and $F^*F = I$. Then the following are equivalent:*

- **(i)** *The anti-Toeplitz operator P_-FP_- is coercive (see* (36)*).*
- **(ii)** $\|\Gamma_F\| < 1$.
- **(iii)** $d(F,\mathcal{H}^\infty_-) < 1$.

If $F \in \mathcal{H}^\infty(\mathcal{X},\mathcal{Y})$, $F^*F = I$, and (ii) holds, then the best possible norm for a left inverse $G \in \mathcal{H}^\infty(\mathcal{Y},\mathcal{X})$ of F is given by $\|G\|^{-2} = 1 - \|\Gamma_F\|^2$.

Proof: 1° Equivalence (ii)⇔(iii) is Corollary 4.5 (Nehari). Since $F^*F = I$, we have

$$\|g\|^2 = \|Fg\|^2 = \|P_-FP_-g\|^2 + \|P_+FP_-g\|^2 \tag{41}$$

for each $g \in \mathcal{H}^2_-(\mathcal{X})$, hence (i) is equivalent to (ii).

2° Assume that (ii) holds and that $F \in \mathcal{H}^\infty$. By (41), we have $\epsilon^2 = 1 - \|\Gamma_F\|^2$ for the maximal ϵ in (36), so the last claim follows from Theorem 4.8. □

By $\mathcal{P}(\mathcal{X})$ we denote the set of (trigonometric) *polynomials* $\mathbb{D} \to \mathcal{X}$ (or $\mathbb{T} \to \mathcal{X}$), i.e., of the functions of the form $\sum_{k=0}^n z^k x_k$, where $x_0, x_1, \ldots, x_n \in \mathcal{X}$. Thus, $g \in \mathcal{P}(\mathbb{C})$ means that $g(z) = \sum_{k=0}^n \alpha_k z^k$ for some $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$, so $f \in \mathcal{P}(\mathbb{C})\mathcal{X}$ means that f is of the form gx for some $x \in \mathcal{X}$ (and n and $\alpha_0, \alpha_1, \ldots$) i.e., it is a one-dimensional polynomial. Obviously, $\mathcal{P}(\mathbb{C})\mathcal{X} \subset \mathcal{P}(\mathcal{X}) \subset \mathcal{H}^p(\mathcal{X}) \subset \mathrm{L}^p(\mathcal{X})$ and $\mathcal{P}(\mathbb{C})\mathcal{X} \subset \mathcal{H}^p(\mathbb{C})\mathcal{X} \subset \mathrm{L}^p(\mathbb{C})\mathcal{X} \subset \mathrm{L}^p(\mathcal{X})$.

We call $F \in \mathcal{H}^\infty(\mathcal{X},\mathcal{Y})$ *inner* if $F^*F = I$ in $\mathrm{L}^\infty_{\text{strong}}(\mathcal{X})$ (recall Proposition 2.4 and the definition above Lemma 3.4). There are several equivalent conditions for a function being inner:[2]

Theorem 4.10 (Inner) *Let $F \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{X},\mathcal{Y})$ and $p \in [1,\infty]$. Then the claims (i)–(viii) are equivalent:*

- **(i)** $F^*F = I$ *(in $\mathrm{L}^\infty_{\text{strong}}$);*
- **(ii)** $\langle x, F^*Fx\rangle_{\mathcal{X}} = \langle x, Ix\rangle_{\mathcal{X}} = \|x\|^2_{\mathcal{X}}$ *a.e. on $\mathbb{T}$ for each $x \in \mathcal{X}$;*
- **(ii')** $\langle Fx', Fx\rangle = \langle x', x\rangle$ *a.e. on $\mathbb{T}$ for every $x, x' \in \mathcal{X}$;*
- **(iii)** $\|Fx\|_{\mathcal{Y}} = \|x\|_{\mathcal{X}}$ *a.e. on $\mathbb{T}$ for each $x \in \mathcal{X}$;*
- **(iv)** $M_F^*M_F = I$ *on $\mathcal{H}^2(\mathcal{X})$.*
- **(v)** $\|Ff\|_{\mathrm{L}^p} = \|f\|_{\mathcal{H}^p}$ *for each $f \in \mathcal{P}(\mathbb{C})\mathcal{X}$.*
- **(vi)** $\|Ff\|_{\mathrm{L}^p} = \|f\|_{\mathrm{L}^p} \le \infty$ *for each measurable $f : \mathbb{T} \to \mathcal{X}$.*
- **(vii)** *There exists a representative $\tilde{F} \in F$ such that $\tilde{F}(z)^*\tilde{F}(z) = I$ for a.e. $z \in \mathbb{T}$.*
- **(viii)** *There exists a representative $\tilde{F} \in F$ such that $\tilde{F}(z)^*\tilde{F}(z) = I$ for each $z \in \mathbb{T}$.*

[2]If $F \in \mathrm{L}^\infty_{\text{strong}}$ and $F^*F = I$, then F is called rigid. Inner functions are assumed to be analytic too.

Moreover, the following hold:

(a) *If F and $\mathcal{V}$ are as in Theorem 3.1(b1), then $F^*F = I$ iff $F_{X,Y}^* F_{X,Y} = I$ for every $(X,Y) \in \mathcal{V}$.*

(b) *If $\mathcal{X}$ is separable, then $F^*F = I$ iff (vii) is satisfied by every $\tilde{F} \in F$, but this is not the case for general $\mathcal{X}$.*

(c) *For any $F : \mathbb{T} \to \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that (v) holds, we have $F \in \mathrm{L}^\infty_{\text{strong}}$ and $F^*F = I$.*

(d) *A function $G : \mathbb{D} \to \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is inner iff $\|Gf\|_{\mathcal{H}^p} = \|f\|_{\mathcal{H}^p}$ for each $f \in \mathcal{P}(\mathbb{C})\mathcal{X}$, or equivalently, for each $f \in \mathcal{H}^p(\mathcal{X})$.*

(e) *If (i) holds, then $\dim \mathcal{X} \le \dim \mathcal{Y}$.*

(The proof is given below Theorem C.1. See Appendix A for $\dim \mathcal{X}$.)

Even in the scalar case ($\mathcal{X} = \mathbb{C} = \mathcal{Y}$) with $F \in \mathcal{H}^\infty$ we may have $F(z)^*F(z) \neq I$ for each $z \in \mathbb{D}$ (e.g., $F(z) = z$). Note also that we may have $F \in \mathcal{H}^\infty$ inner and $Ff \in \mathcal{H}^\infty$ without f being holomorphic, hence nor $\mathcal{H}^\infty$ (e.g., $F(z) = z$, $f(z) = z^{-1}$ a.e.).

To illustrate (b), in Example C.3 we construct an inner function $F \in \mathcal{H}^\infty(\ell^2(\mathbb{T}), \ell^2(\mathbb{T}))$ for which any representative $\tilde{F} \in F$ satisfying (vii) or (viii) is artificial. For this F there exists a unique "natural" boundary function $\tilde{F} \in \mathrm{L}^\infty_{\text{strong}}$ of F, namely the strong limit of F everywhere on $\mathbb{T}$, and that function has $\tilde{F}(z)^*\tilde{F}(z) \neq I$ for each $z \in \mathbb{T}$.

The claim in (d) does not hold for every holomorphic $f : \mathbb{D} \to \mathcal{X}$. E.g., if we define the inner function $G \in \mathcal{H}^\infty(\mathbb{C})$ by $G(z) := \mathrm{e}^{(z+1)/(z-1)}$, then $G^{-1}(z) = \mathrm{e}^{(1+z)/(1-z)} \notin \mathcal{H}^\infty$, hence $f := G^{-1}g \notin \mathcal{H}^2(\mathbb{C})$ for some $g \in \mathcal{H}^2$ although $Gf \in \mathcal{H}^2$, by Theorem C.1.

We record an important special case of the last claim in Theorem 4.9:

Corollary 4.11 (Coprime) *Let $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2$ be Hilbert spaces. If a map $\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right] \in \mathcal{H}^\infty(\mathcal{X}, \mathcal{Y}_1 \times \mathcal{Y}_2)$ is inner, then F and G are right coprime iff $\|\Gamma_{\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]}\| < 1$.*

Functions F and G being *right coprime* means that $\tilde{F}F + \tilde{G}G = I$ on $\mathbb{D}$ for some $\begin{bmatrix} \tilde{F} & \tilde{G} \end{bmatrix} \in \mathcal{H}^\infty(\mathcal{Y}_1 \times \mathcal{Y}_2, \mathcal{X})$. In systems theory, a "right fraction" FG^{-1} is called "normalized" iff $\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ is inner (i.e., iff $F^*F + G^*G = I$ in $\mathrm{L}^\infty_{\text{strong}}$) [CO06] [Mik07b].

Recall that for $F \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{X}, \mathcal{Y})$, $f \in \mathcal{H}^2(\mathcal{X})$ we have set $(M_F f)(z) := F(z)f(z)$ $(z \in \mathbb{T})$; if $F \in \mathcal{H}^\infty$, then $(M_F f)(z) = F(z)f(z)$ $(z \in \mathbb{D})$ too, by Proposition 2.4. Analogously, we define its restriction $N_F : \mathcal{H}^2(\mathcal{X}) \to \mathcal{H}^2(\mathcal{Y})$ by $N_F f := Ff$. It is known that if F^* is holomorphic, then F is a constant:

Lemma 4.12 (Causal and anticausal) *Let $F \in \mathcal{H}^\infty(\mathcal{X},\mathcal{Y})$. Then the following are equivalent:*

(i) $F^* \in \mathcal{H}^\infty(\mathcal{Y},\mathcal{X})$*, where* $F^* : \mathbb{D} \ni z \mapsto F(z)^*$*;*

(i') $[F]^* \in \mathcal{H}^\infty(\mathcal{Y},\mathcal{X})$ *(i.e., the class* $[F]^* \in \mathrm{L}^\infty_{\mathrm{strong}}$ *is the boundary function of some* $G \in \mathcal{H}^\infty(\mathcal{Y},\mathcal{X})$*);*

(ii) $M_F^* = M_G$ *for some* $G \in \mathcal{H}^\infty$*;*

(iii) $N_F^* = N_G$ *for some* $G \in \mathcal{H}^\infty$*;*

(iv) F *is a constant function (i.e.,* $F \in \mathcal{B}(\mathcal{X},\mathcal{Y})$*).*

Proof: The equivalence of (iii) and (iv) follows from Theorem 1.15B, p. 15 of [RR85]. By the definition of $[F]^*$ (above Lemma 3.4), (i') and (iii) are equivalent. The implications (iv)$\Rightarrow$(i)$\Rightarrow$(iii), and (iv)$\Rightarrow$(ii)$\Rightarrow$(iii) are obvious. □

The first and last of the following "well-known" results are often phrased as "$\mathcal{H}^\infty$ consists of the (causal) shift-invariant maps on $\mathcal{H}^2$", and as "inner-outer means constant":

Proposition 4.13 (Causal; inner-outer) *Let $T \in \mathcal{B}(\mathcal{H}^2(\mathcal{X}),\mathcal{H}^2(\mathcal{Y}))$.*
Then $T = N_F$ for some $F \in \mathcal{H}^\infty(\mathcal{X},\mathcal{Y})$ iff $ST = TS$.
Let $F \in \mathcal{H}^\infty(\mathcal{X},\mathcal{Y})$.
Then $F^{-1} \in \mathcal{H}^\infty(\mathcal{Y},\mathcal{X})$ iff N_F is invertible $\mathcal{H}^2(\mathcal{X}) \to \mathcal{H}^2(\mathcal{Y})$.
If F is inner and $\overline{N_F[\mathcal{H}^2(\mathcal{X})]} = \mathcal{H}^2(\mathcal{Y})$, then $F = F^{-*} \in \mathcal{B}(\mathcal{X},\mathcal{Y})$.

Proof of Proposition 4.13: 1° The first claim is from Theorem 1.15B, p. 15 of [RR85]. There $\mathcal{X} = \mathcal{Y}$ is assumed, but one can consider T (with zero extension) as an operator $\mathcal{H}^2(\mathcal{X}\times\mathcal{Y}) \to \mathcal{H}^2(\mathcal{X}\times\mathcal{Y})$ and then remove the zero extension of F.

2° If F is invertible in $\mathcal{H}^\infty$, then, on $\mathcal{H}^2$ we have $N_F N_{F^{-1}} = I$ and $N_{F^{-1}} N_F = I$ (Proposition 2.4), so assume that N_F^{-1} exists. Now

$$N_F^{-1} S f = N_F^{-1} S N_F N_F^{-1} f = N_F^{-1} N_F S N_F^{-1} f = S N_F^{-1} f \ \forall f \in \mathcal{H}^2(\mathcal{Y}), \quad (42)$$

hence $N_F^{-1} = N_G$ for some $G \in \mathcal{H}^\infty(\mathcal{Y},\mathcal{X})$, by 1°. Obviously, we must have $G(z) = F(z)^{-1} \ \forall z \in \mathbb{D}$.

3° Assume now that F is inner (so $N_F^* N_F = I$). Then $N_F[\mathcal{H}^2(\mathcal{X})]$ is closed, hence then $\overline{N_F[\mathcal{H}^2(\mathcal{X})]} = \mathcal{H}^2(\mathcal{Y})$ implies that N_F is invertible (hence unitary), so $F^{-1} \in \mathcal{H}^\infty$, by 2°. But $N_{F^{-1}} = N_F^* N_F N_{F^{-1}} = N_F^*$, so $F \in \mathcal{B}$, by Lemma 4.12, hence $F^* = F^{-1}$. □

"Has smaller range than" means "is divisible by":

Theorem 4.14 (Divisor) *Assume that $F \in \mathcal{H}^\infty(\mathcal{X},\mathcal{Y})$, $G \in \mathcal{H}^\infty(\mathcal{Z},\mathcal{Y})$ for some Hilbert space $\mathcal{Z}$, and G is inner. Then $M_F[\mathcal{H}^2(\mathcal{X})] \subset M_G[\mathcal{H}^2(\mathcal{Z})]$ iff G is a left divisor of F.*

The latter means that $F = GK$ for some $K \in \mathcal{H}^\infty(\mathcal{X}, \mathcal{Z})$. If also F is inner, then so is K.

Proof: "If" is obvious, so assume that $M_F[\mathcal{H}^2(\mathcal{X})] \subset M_G[\mathcal{H}^2(\mathcal{Z})]$. Then, for each $f \in \mathcal{H}^2(\mathcal{X})$ there exists a unique $g_f \in \mathcal{H}^2(\mathcal{Z})$ such that $M_G g_f = M_F f$; left T denote the map $f \mapsto g_f$. Then $M_F = M_G T$, $\|T\| \le \|F\|$, $T : \mathcal{H}^2(\mathcal{X}) \to \mathcal{H}^2(\mathcal{Z})$ is linear, and $M_F S f = S M_F f = S M_G T f = M_G S T f$, hence $g_{Sf} = STf$, i.e., $TSf = STf$, for every $f \in \mathcal{H}^2(\mathcal{X})$. By Proposition 4.13, $T \in \mathcal{H}^\infty(\mathcal{X}, \mathcal{Y})$. □

We call $\mathcal{M} \subset \mathcal{H}^2(\mathcal{X})$ *shift-invariant* if $S\mathcal{M} = \mathcal{M}$. Such subspaces are ranges of "unique" inner functions:

Theorem 4.15 (Lax–Halmos) *A closed subspace $\mathcal{M}$ of $\mathcal{H}^2(\mathcal{X})$ is shift-invariant iff $\mathcal{M} = M_F[\mathcal{H}^2(\mathcal{X}_0)]$ for some closed subspace $\mathcal{X}_0 \subset \mathcal{X}$ and some inner $F \in \mathcal{H}^\infty(\mathcal{X}_0, \mathcal{X})$.*

If also $\mathcal{M} = M_G[\mathcal{H}^2(\mathcal{X}_1)]$ for some Hilbert space $\mathcal{X}_1$ and some inner $G \in \mathcal{H}^\infty(\mathcal{X}_1, \mathcal{X})$, then $G = FT$ for some $T = T^{-*} \in \mathcal{B}(\mathcal{X}_1, \mathcal{X}_0)$.

Proof: 1° *Existence:* "If" is obvious ($SM_F g = M_F S g \in M_F[\mathcal{H}^2(\mathcal{X}_0)]$ for each $g \in \mathcal{H}^2(\mathcal{X}_0)$), so we only prove "only if". Let $\mathcal{V}$ be as in Theorem 3.2 (with $0 : \mathbb{D} \to \mathcal{B}(\mathcal{X}, \mathcal{X})$ in place of F, because we just need a complete collection of separable orthogonal subspaces). For each $(X, X) \in \mathcal{V}$, the subspace $\mathcal{M}_X := \mathcal{M} \cap \mathcal{H}^2(X)$ is closed and shift-invariant, hence $\mathcal{M}_X = F_X[\mathcal{H}^2(\mathcal{X}_X)]$ for some closed $\mathcal{X}_X \subset X$ and some inner $F_X \in \mathcal{H}^\infty(\mathcal{X}_X, X)$, by the separable case of this theorem (e.g., p. 17 of [Nik86]).

Define $\mathcal{Z} \subset \prod_{(X,X)\in\mathcal{V}} \mathcal{X}_X$ as in Lemma A.3. Then $F := \sum P_X^* F_X P_{\mathcal{X}_X} \in \mathcal{H}^\infty(\mathcal{Z}, \mathcal{X})$ is inner, by Theorem 3.1(a2) (we have a priori $\|F\|_{\mathcal{H}^\infty} \le 1 < \infty$ because $\|F_X\|_{\mathcal{H}^\infty} \le 1$, for each X, because F_X is inner, hence (a2) is applicable). Given $g \in \mathcal{M}$ and $(X, X) \in \mathcal{V}$, we have $P_X g \in \mathcal{M}_X$, hence $P_X g = F_X f_X$ for some $f_X \in \mathcal{H}^2(\mathcal{X}_X)$. But $\|f_X\|_2 = \|P_X g\|_2$, because F_X is inner. By Theorem 3.1(a1), $f := \sum P_X^* f_X \in \mathcal{H}^2(\mathcal{X})$, $\|f\|_2 = \|g\|_2$, and

$$Ff = \sum P_X^* F_X f = \sum P_X^* P_X g = \sum \tilde{P}_X g = g. \tag{43}$$

Thus, $\mathcal{M} \subset F[\mathcal{H}^2(\mathcal{Z})]$. Conversely, given $f \in \mathcal{H}^2(\mathcal{Z})$, we have $Ff = \sum P_X^* F_X P_X f \in \mathcal{M}$, by Theorem 3.1(a1), hence $\mathcal{M} = F[\mathcal{H}^2(\mathcal{Z})]$.

By Lemma A.2, the space $\mathcal{Z}$ is unitarily equivalent to a closed subspace $\mathcal{X}_0$ of $\mathcal{X}$, so we can replace F by FT^{-1}, where $T = T^{-*} \in \mathcal{B}(\mathcal{Z}, \mathcal{X}_0)$, because $FT^{-1}[\mathcal{H}^2(\mathcal{X}_0)] = F[\mathcal{H}^2(\mathcal{Z})] = \mathcal{M}$.

2° *Uniqueness:* By Theorem 4.14, $G = FT$, where $T \in \mathcal{H}^\infty(\mathcal{X}_1, \mathcal{X}_0)$. But T is inner and M_T is onto, hence $T = T^{-*} \in \mathcal{B}(\mathcal{X}_1, \mathcal{X}_0)$, by Proposition 4.13.

□

We say that $\mathcal{M} \subset \mathcal{H}^2(\mathcal{X})$ *reduces the shift* if $S\mathcal{M} \subset \mathcal{M}$ and $P_+ S^* \mathcal{M} \subset \mathcal{M}$.

Theorem 4.16 (Reducing subspace) *A closed subspace $\mathcal{M}$ of $\mathcal{H}^2(\mathcal{X})$ reduces the shift iff $\mathcal{M} = \mathcal{H}^2(\mathcal{X}_0)$ for some closed subspace $\mathcal{X}_0 \subset \mathcal{X}$.*

Proof: Set $\mathcal{X}_0' := \cup\{f[\mathbb{D}] \bigm| f \in \mathcal{M}\}$, $\mathcal{X}_0 := \overline{\mathcal{X}_0'}$. Then $\mathcal{X}_0$ is a closed subspace of $\mathcal{X}$ and $\mathcal{M} \subset \mathcal{H}^2(\mathcal{X}_0)$.

We want to show that $\mathcal{M} = \mathcal{H}^2(\mathcal{X}_0)$, so to get a contradiction, we assume that there exists $g \in \mathcal{H}^2(\mathcal{X}_0) \setminus \mathcal{M}$. Since $M' := g[\mathbb{D}] \subset \mathcal{X}_0$ is separable (because g is continuous and $\mathbb{D}$ is separable), so is $M := \overline{M'} \subset \mathcal{X}_0$. Therefore, there exist $\{f_1, f_2, \ldots\} \subset \mathcal{M}$ such that

$$M \subset \overline{\cup_{k=1}^{\infty} f_k[\mathbb{D}]} =: \mathcal{X}_1 \tag{44}$$

(to prove this, let $S \subset M$ be countable and dense and for each $h \in S$ and $j \in \mathbb{Z}_+$ choose $f_{h,j} \in \mathcal{M}$ and $z_{h,j} \in \mathbb{D}$ so that $\|f_{h,j}(z_{h,j}) - h\| < 1/j$). Then $g \in \mathcal{H}^2(M) \subset \mathcal{H}^2(\mathcal{X}_1)$.

But $\mathcal{M} \cap \mathcal{H}^2(\mathcal{X}_1)$ reduces S and $\mathcal{X}_1$ is separable, hence $\mathcal{M} \cap \mathcal{H}^2(\mathcal{X}_1) = \mathcal{H}^2(\mathcal{X}_2)$ for some closed subspace $\mathcal{X}_2 \subset \mathcal{X}_1$ (by the separable case of Theorem 4.16; see Corollary on p. 96 of [RR85]). But $f_k \in \mathcal{M} \cap \mathcal{H}^2(\mathcal{X}_1) = \mathcal{H}^2(\mathcal{X}_2)$ for each k, hence $\mathcal{X}_1 \subset \mathcal{X}_2$, hence $\mathcal{X}_1 = \mathcal{X}_2$, hence $\mathcal{M} \cap \mathcal{H}^2(\mathcal{X}_1) = \mathcal{H}^2(\mathcal{X}_1) \ni g$, a contradiction. $\square$

A map $F \in \mathcal{H}^2_{\text{strong}}(\mathcal{X}, \mathcal{Y})$ is called *outer* if the set $\{Fp \bigm| p \in \mathcal{P}(\mathcal{X})\}$ is dense in $\mathcal{H}^2(\mathcal{Y})$. (Recall that $\mathcal{P}(\mathcal{X})$ is the set of polynomials, i.e., of functions of the form $\sum_{k=0}^{n} z^k x_k$.)

Thus, if $F \in \mathcal{H}^\infty(\mathcal{X}, \mathcal{Y})$, then F is outer iff $M_F[\mathcal{H}^2(\mathcal{X})]$ is dense in $\mathcal{H}^2(\mathcal{Y})$, because M_F is then continuous $\mathcal{H}^2 \to \mathcal{H}^2$.

Theorem 4.17 (Inner-Outer Factorization) *Every $F \in \mathcal{H}^2_{\text{strong}}(\mathcal{X}, \mathcal{Y})$ can be expressed as $F = F_i F_o$, where $F_o \in \mathcal{H}^2_{\text{strong}}(\mathcal{X}, \mathcal{Y}_0)$ is outer and $F_i \in \mathcal{H}^\infty(\mathcal{Y}_0, \mathcal{Y})$ is inner, $\mathcal{Y}_0$ being a closed subspace of $\mathcal{Y}$. Moreover, $\|F_o\|_{\mathcal{H}^2_{\text{strong}}} = \|F\|_{\mathcal{H}^2_{\text{strong}}}$, $\|F_o\|_{\mathcal{H}^\infty} = \|F\|_{\mathcal{H}^\infty} \le \infty$, and $\dim \mathcal{Y}_0 \le \dim \mathcal{X}$.*

If also $F = F_i' F_o'$, where $F_o' \in \mathcal{H}^2_{\text{strong}}(\mathcal{X}, \mathcal{Z}')$ is outer and $F_i' \in \mathcal{H}^\infty(\mathcal{Z}', \mathcal{Y})$ is inner, $\mathcal{Z}'$ being a Hilbert space, then there exists $T = T^{-*} \in \mathcal{B}(\mathcal{Z}', \mathcal{Y}_0)$ such that $F_i' = F_i T$ and $F_o' = T^* F_o$.

(Because F_i is inner, we have $\|F_o\| = \|F\|$ for almost any reasonable norm.)

Proof: Also this could be deduced from the separable case. However, we shall deduce this from Theorem 4.15.

Since $M_F[\mathcal{P}(\mathcal{X})]$ is a shift-invariant subspace of $\mathcal{H}^2(\mathcal{Y})$, so is its closure, which equals $M_{F_i}[\mathcal{H}^2(\mathcal{Y}_0)]$ for some closed subspace $\mathcal{Y}_0 \subset \mathcal{Y}$ and some inner $F_i \in \mathcal{H}^\infty(\mathcal{Y}_0, \mathcal{X})$, by Theorem 4.15. For each $x \in \mathcal{X}$, there exists a unique $f_x \in \mathcal{H}^2(\mathcal{Y}_0)$ such that $Fx = M_{F_i} f_x$. The map $T : x \mapsto f_x$ is linear, hence so is $F_o(z) : x \mapsto f_x(z)$, for each $z \in \mathbb{D}$. By [HP57, Theorem 3.10.1], $F_o : \mathbb{D} \to \mathcal{B}(\mathcal{X}, \mathcal{Y}_0)$ is holomorphic. Obviously, $\|f_x\|_2 = \|Fx\|_2$ for every x, hence $\|F_o\|_{\mathcal{H}^2_{\text{strong}}} = \|F\|_{\mathcal{H}^2_{\text{strong}}}$. By the continuity of M_{F_i}, we have

$$\overline{M_{F_i} \overline{M_{F_o}[\mathcal{P}(\mathcal{X})]}}, = \overline{M_{F_i} M_{F_o}[\mathcal{P}(\mathcal{X})]} = \overline{M_F[\mathcal{P}(\mathcal{X})]} = M_{F_i}[\mathcal{H}^2(\mathcal{Y}_0)] \tag{45}$$

hence the function F_o must be outer.

By Theorem 4.15, $F_i' = F_i T$ for any other inner-outer factorization $F = F_i' F_o'$ of F. But then, for $z \in \mathbb{D}$, we have $T^* F_o(z) = T^* F_i(z)^* F(z) = (F_i'(z))^* F(z) = F_o'(z)$.

3° $\dim \mathcal{Y}_0 \leq \dim \mathcal{X}$*:* Because $\mathcal{H}^2(\mathcal{Y}) \ni g \mapsto g(0)$ is bounded, $F_0(0)[\mathcal{X}] \subset \mathcal{Y}_0$ must be dense, so $\dim \mathcal{X} \geq \dim \mathcal{Y}_0$, by Lemma A.1(a). $\square$

Assume that F, F_o and F_i are as above and $F \in \mathcal{H}^\infty$. Then $F^* F = F_o^* F_o$ in $\mathrm{L}^\infty_{\text{strong}}$. Moreover, $M_F^* M_F \geq \epsilon I$ for some $\epsilon > 0$ (i.e., F is left-invertible in $\mathrm{L}^\infty_{\text{strong}}$) iff F_o is invertible in $\mathcal{H}^\infty$. If it is, then F_o is called a (invertible) *spectral factor* of $F^* F$. (All this is well known and the claims follow easily from the above.)

5 Results for the real line

In this section we present results analogous to those in the previous sections but with the real line $\mathbb{R}$ (resp., half-plane $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$) in place of the unit circle $\mathbb{T}$ (resp., disc $\mathbb{D}$). The main difference is that we want to use the translations $\tau^t : f \mapsto f(t + \cdot)$ instead of the right-shift S. The *half-plane notation* used in this section differs from the *disc notation* of the previous sections.

We first state that the results in the previous sections hold with this notation too (Lemma 5.1). Then we rewrite those corresponding to Section 4 to their "time-domain" forms, using the "Fourier multiplier" result that the elements of $\mathrm{L}^\infty_{\text{strong}}$ (resp., $\mathcal{H}^\infty$) correspond isometrically to the time-invariant (resp., causal) operators $\mathrm{L}^2(\mathcal{X}) \to \mathrm{L}^2(\mathcal{Y})$ (Theorem 5.2). One easily verifies that such "time-domain" forms could be used in discrete time too, on operators $\ell^2(\mathcal{X}) \to \ell^2(\mathcal{Y})$ (cf. Remarks 2.1). Most comments and explaining text in Section 4 applies here too. The proofs are given in Section 6.

We start by presenting some of this half-plane notation. Let B be a Banach space and let $1 \leq p \leq \infty$. By $\mathrm{L}^p(B)$ we denote the L^p space of functions $\mathbb{R} \to B$. By $\mathcal{H}^p(B)$ we denote the Banach space of holomorphic functions $f : \mathbb{C}^+ \to B$ for which

$$\|f\|_{\mathcal{H}^p} := \sup_{r>0} \|f(\cdot + ir)\|_{\mathrm{L}^p} < \infty. \tag{46}$$

Moreover, $\mathcal{H}^\infty_-(B)$ stands for the Banach space of bounded holomorphic functions $\mathbb{C}^- \to B$, where $\mathbb{C}^- := \{\operatorname{Im} z < 0\}$. By P_+ we denote the orthogonal projection $\mathrm{L}^2 \to \mathcal{H}^2$ for any Hilbert space H. Again $P_- := I - P_+$, $\mathcal{H}^2_- := P_-[\mathrm{L}^2]$. The $\mathrm{L}^p_{\text{strong}}$ and $\mathcal{H}^p_{\text{strong}}$ spaces are defined as before (now on $\mathbb{R}$ and on $\mathbb{C}^+$, respectively).

We now record the fact that all above results hold with this half-plane notation too (the remaining definitions will follow).

Lemma 5.1 *Propositions 2.2– 2.4 hold with this notation too except that we must replace* $1-$ *(resp.,* $r\cdot$*,* rz*) by* $0+$ *(resp.,* $\cdot + ir$*,* $z + ir$*), and that the*

Poisson integral is different [RR85] [Mik08]. Also the results in Section 3 hold with this notation.

The results in Section 4 hold with this notation too except that we omit Theorem 4.7(iv) and reformulate Proposition 4.13 and the definitions above Theorems 4.15–4.17 as given below (see Proposition 5.12, Theorems 5.14–5.16 and Lemma 5.17) and that $\mathcal{P}$ must be replaced by $\tilde{\mathcal{P}}_p$, which will be defined below Remark 6.2.

In all above results, we assume that $\mathbb{T}$ (resp., $\mathbb{D}$, $\mathbb{D}^-$) has been replaced by $\mathbb{R}$ (resp., $\mathbb{C}^+$, $\mathbb{C}^-$).

(The proof is given in Lemma 6.3. Alternative explicit versions of the results in Section 4 are given below. See above Theorem 5.7 for $\mathcal{C}(\mathcal{BC}(\mathcal{X},\mathcal{Y}))$.)

Next we present the *time-domain* concepts corresponding to those above. Analogous concepts also exist in discrete time (corresponding to the "disc notation") but they are more useful in continuous time, since the translations τ^t do not have such nice "frequency-domain" (Fourier/Laplace side) equivalents.

By $\mathrm{TI}(\mathcal{X},\mathcal{Y})$ we denote the operators $\mathscr{E} : \mathrm{L}^2(\mathcal{X}) \to \mathrm{L}^2(\mathcal{Y})$ that are *translation-invariant*: $\mathscr{E}\tau^t = \tau^t\mathscr{E}$ for every $t \in \mathbb{R}$ (or equivalently, for every $t \in (0,1)$).

We set $\pi_+ f := \begin{cases} f(t), & t \geq 0; \\ 0, & t < 0 \end{cases}$, $\pi_- := I - \pi_+$. We identify any function f defined on $\mathbb{R}_+ := [0,\infty)$ with its zero extension to $\mathbb{R}$. By $\mathrm{L}^2_+(\mathcal{X})$ we denote the Hilbert space of $\mathrm{L}^2(\mathcal{X})$ functions supported on $\mathbb{R}_+$, and by $\mathrm{L}^2_-(\mathcal{X})$ the orthogonal complement of $\mathrm{L}^2(\mathcal{X})$.

By $\mathrm{TIC}(\mathcal{X},\mathcal{Y})$ we denote the operators $\mathscr{D} \in \mathrm{TI}(\mathcal{X},\mathcal{Y})$ that are *causal*: $\pi_-\mathscr{D}\pi_+ = 0$. Both $\mathrm{TI}(\mathcal{X},\mathcal{Y})$ and $\mathrm{TIC}(\mathcal{X},\mathcal{Y})$ are obviously closed subspaces of $\mathcal{B}(\mathrm{L}^2(\mathcal{X}),\mathrm{L}^2(\mathcal{Y}))$; whose norm we use.

By $\mathcal{F}f := \widehat{f}(s) := \int_{-\infty}^{\infty} \mathrm{e}^{-ist} f(t)\,dt$ we denote the Fourier–Laplace transform of a function for those $s \in \mathbb{C}$ for which $\widehat{f}(s)$ converges absolutely. If $f \in \mathrm{L}^1$, then $\widehat{f} \in \mathrm{L}^\infty$. This extends to a unitary map $\mathrm{L}^2(\mathcal{X}) \to \mathrm{L}^2(\mathcal{X})$ satisfying $\mathcal{F}\pi_+ = P_+\mathcal{F}$. Thus, if $f \in \mathrm{L}^2_+(\mathcal{X})$, then $\widehat{f} \in \mathcal{H}^2(\mathcal{X})$, and $\widehat{f}_{|\mathbb{R}}$ coincides with the boundary function of $\widehat{f}_{|\mathbb{C}^+}$.

Now we recall the half-plane form of Theorem 4.3 from [Mik08].

Theorem 5.2 **($\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$)** *For each $\mathscr{E} \in \mathrm{TI}(\mathcal{X},\mathcal{Y})$ there exists a unique function (equivalence class) $\hat{\mathscr{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ such that $\hat{\mathscr{E}}\widehat{f} = \widehat{\mathscr{E}f}$ a.e. on $\mathbb{R}$ for every $f \in \mathrm{L}^2(\mathcal{X})$. Moreover, $\|\hat{\mathscr{E}}\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \|\mathscr{E}\|_{\mathcal{B}(\mathrm{L}^2(\mathcal{X}),\mathrm{L}^2(\mathcal{Y}))}$, and every $\hat{\mathscr{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ is of this form.*

The following is well known [Wei91]:

Proposition 5.3 **($\widehat{\mathrm{TIC}} = \mathcal{H}^\infty$)** *For any $\mathscr{D} \in \mathrm{TIC}(\mathcal{X},\mathcal{Y})$ there exists a unique function $\hat{\mathscr{D}} \in \mathcal{H}^\infty(\mathcal{X},\mathcal{Y})$ such that $(\widehat{\mathscr{D}f})(z) = \hat{\mathscr{D}}(z)\widehat{f}(z)$ for all $z \in \mathbb{C}^+$ and all $f \in \mathrm{L}^2_+(\mathcal{X})$.*

Moreover, this identification is an isometric isomorphism of TIC *onto* $\mathcal{H}^\infty$.

Naturally, the strong boundary function $\lim_{r\to 0+}\hat{\mathscr{D}}(\cdot + ir)$ equals that given by Theorem 5.2, analogously to Proposition 2.4. We identify $\hat{\mathscr{D}}|_{\mathbb{C}^+}$ with $\hat{\mathscr{D}}|_{\mathbb{R}}$.

The operators $\Gamma \in \mathcal{B}(\mathrm{L}^2_-(\mathcal{X}), \mathrm{L}^2(\mathcal{Y}))$ that satisfy $\pi_+\tau^t\Gamma = \Gamma\tau^t\pi_-$ $(t > 0)$ are called *Hankel operators*. So they are "left-translation-invariant" maps $\mathrm{L}^2_- \to \mathrm{L}^2_+$ in certain sense.

The *Hankel operator* $\Gamma_{\mathscr{E}} \in \mathcal{B}(\mathrm{L}^2_-(\mathcal{X}), \mathrm{L}^2(\mathcal{Y}))$ *of* $\mathscr{E} \in \mathrm{TI}(\mathcal{X}, \mathcal{Y})$ is defined by

$$\Gamma_{\mathscr{E}} := \pi_+\mathscr{E}\pi_-. \tag{47}$$

As in (24), one can verify that all such operators are Hankel operators. As mentioned in Lemma 5.1, Theorem 4.4 holds in this half-plane notation too. Below we give its time-domain form.

Theorem 5.4 *An operator* $\Gamma : \mathrm{L}^2_-(\mathcal{X}) \to \mathrm{L}^2(\mathcal{Y})$ *is a Hankel operator iff* $\Gamma = \Gamma_{\mathscr{E}}$ *for some* $\mathscr{E} \in \mathrm{TI}(\mathcal{X}, \mathcal{Y})$. *If* Γ *is a Hankel operator, then we can choose* $\mathscr{E}$ *so that* $\Gamma = \Gamma_{\mathscr{E}}$ *and* $\|\mathscr{E}\| = \|\Gamma\|$. *Finally, if* $\mathscr{E}, \mathscr{G} \in \mathrm{TI}(\mathcal{X}, \mathcal{Y})$, *then* $\Gamma_{\mathscr{E}} = \Gamma_{\mathscr{G}}$ *iff* $\mathscr{E}^* - \mathscr{G}^* \in \mathrm{TIC}(\mathcal{Y}, \mathcal{X})$.

Note that $\mathscr{E}^* - \mathscr{G}^* \in \mathrm{TIC}$ iff $\pi_+(\mathscr{E} - \mathscr{G})\pi_- = 0$.

Corollary 5.5 (Nehari–Page) *For any* $\mathscr{E} \in \mathrm{TI}(\mathcal{X}, \mathcal{Y})$ *we have*

$$\|\Gamma_{\mathscr{E}}\| = \min_{\mathscr{D} \in \mathrm{TIC}(\mathcal{Y}, \mathcal{X})} \|\mathscr{E} - \mathscr{D}^*\|. \tag{48}$$

Theorem 5.6 (Adamjan–Arov–Krein) *For any* $\mathscr{E} \in \mathrm{TI}(\mathcal{X}, \mathcal{Y})$ *and* $n \in \{1, 2, \ldots\}$ *we have*

$$s_n(\Gamma_{\mathscr{E}}) = \min_{\mathscr{G} \in \mathrm{TI}} \{\|\mathscr{E} - \mathscr{G}\| \mid \operatorname{rank} \Gamma_{\mathscr{G}} \le n - 1\}. \tag{49}$$

By $\mathcal{C}(\mathcal{BC}(\mathcal{X}, \mathcal{Y}))$ we denote the set of continuous functions $F : \mathbb{R} \to \mathcal{BC}(\mathcal{X}, \mathcal{Y})$ for which $\lim_{t\to\pm\infty} F(t)$ exists; here $\mathcal{BC}$ stands for the set of compact linear operators $\mathcal{X} \to \mathcal{Y}$.

Theorem 5.7 (Hartman) *Let* $\mathscr{E} \in \mathrm{TI}(\mathcal{X}, \mathcal{Y})$. *Then the following are equivalent:*

- **(i)** $\Gamma_{\mathscr{E}}$ *is compact* $\mathrm{L}^2_-(\mathcal{X}) \to \mathrm{L}^2(\mathcal{Y})$;
- **(ii)** $\hat{\mathscr{E}} \in \mathcal{H}^\infty_-(\mathcal{X}, \mathcal{Y}) + \mathcal{C}(\mathcal{BC}(\mathcal{X}, \mathcal{Y}))$;
- **(iii)** $\Gamma_{\mathscr{E}} = \Gamma_{\mathscr{G}}$ *for some* $\hat{\mathscr{G}} \in \mathcal{C}(\mathcal{BC}(\mathcal{X}, \mathcal{Y}))$;

If (iii) holds and $\epsilon > 0$, *then we can choose the* $\mathscr{G}$ *in (iii) so that* $\|\mathscr{G}\| < \|\Gamma_{\mathscr{E}}\| + \epsilon$ *and* $\mathscr{G} = \tilde{P}_Y\mathscr{G}\tilde{P}_X$ *for some closed separable subspaces* $X \subset \mathcal{X}$ *and* $Y \subset \mathcal{Y}$.

Condition (iv) of Theorem 4.7 could be written as "$A_{\mathscr{E}} = A_{\mathscr{G}}$ on $\mathbb{R}_+$ for some $\hat{\mathscr{G}} \in \mathcal{C}(\mathcal{BC}(\mathcal{X}, \mathcal{Y}))$", where $A_{\mathscr{E}}$ is the distribution whose Fourier transform equals $\hat{\mathscr{E}}$ (so $\mathscr{E} f = A_{\mathscr{E}} * f$).

The reflection $\mathcal{R}$ is defined by $(\mathcal{R}f)(t) := f(-t)$. On (i') below note that $\mathscr{E}^{\mathrm{d}} \in \mathrm{TI}$, $\mathscr{D}^{\mathrm{d}} \in \mathrm{TIC}$, $(\mathscr{E}^{\mathrm{d}})^{\mathrm{d}} = \mathscr{E}$ and $\mathcal{F}(\mathscr{E}^{\mathrm{d}}) = \hat{\mathscr{E}}(\bar{\cdot})^*$ for every $\mathscr{E} \in \mathrm{TI}$, $\mathscr{D} \in \mathrm{TIC}$.

Theorem 5.8 (Tolokonnikov) *Let $\mathscr{D} \in \mathrm{TIC}(\mathcal{X}, \mathcal{Y})$. Then the following are equivalent:*

- **(i)** *The anti-Toeplitz operator $\pi_- \mathscr{D} \pi_-$ is* coercive, *i.e., there exists $\epsilon > 0$ such that for each $g \in \mathrm{L}^2_-(\mathcal{X})$ we have*

$$\|\pi_- \mathscr{D} \pi_- g\|_2 \geq \epsilon \|g\|_2. \tag{50}$$

- **(i')** *The map $\mathscr{D}^{\mathrm{d}} := \mathcal{R}\mathscr{D}^*\mathcal{R}$ maps $\mathrm{L}^2_+(\mathtt{Y})$ onto $\mathrm{L}^2_+(\mathtt{X})$.*
- **(ii)** *$\mathscr{G}\mathscr{D} = I$ for some $\mathscr{G} \in \mathrm{TIC}(\mathcal{Y}, \mathcal{X})$.*
- **(iii)** *There exist a closed subspace $\mathcal{Z} \subset \mathcal{Y}$ and a map $\tilde{\mathscr{D}} \in \mathrm{TIC}(\mathcal{Z}, \mathcal{X})$ such that $\begin{bmatrix} \mathscr{D} & \tilde{\mathscr{D}} \end{bmatrix} \in \mathrm{TIC}(\mathcal{X} \times \mathcal{Z}, \mathcal{Y})$ is invertible.*

Assume (i). Then the best possible norm of a left-inverse $\mathscr{G}$ in (ii) is $1/\epsilon$ for the maximal ϵ in (50). Set $M := \|\mathscr{D}\|$. In (iii), (if $\mathcal{X} \neq \{0\}$) we can have $\|\tilde{\mathscr{D}}\| = 1$, $\| \begin{bmatrix} \mathscr{D} & \tilde{\mathscr{D}} \end{bmatrix} \| \leq \sqrt{M^2 + 1}$, and

$$\| \begin{bmatrix} \mathscr{D} & \tilde{\mathscr{D}} \end{bmatrix}^{-1} \|_{\mathcal{H}^\infty} \leq \frac{M}{\epsilon}\sqrt{1 + \epsilon^{-2}}. \tag{51}$$

If, in Theorem 5.8, we have $\mathscr{D}^*\mathscr{D} = I$, then one more equivalent condition is that the Hankel norm of $\mathscr{D}$ is less than one.

Theorem 5.9 *Let $\mathscr{E} \in \mathrm{TI}(\mathcal{X}, \mathcal{Y})$ and $\mathscr{E}^*\mathscr{E} = I$. Then the following are equivalent:*

- **(i)** *The anti-Toeplitz operator $\pi_- \mathscr{E} \pi_-$ is coercive.*
- **(ii)** *$\|\Gamma_{\mathscr{E}}\| < 1$.*
- **(iii)** *$d(\mathscr{E}^*, \mathrm{TIC}) < 1$.*

If $\mathscr{E} \in \mathcal{H}^\infty(\mathcal{X}, \mathcal{Y})$, $\mathscr{E}^*\mathscr{E} = I$, and (ii) holds, then the best possible norm for a left inverse $\mathscr{G} \in \mathrm{TIC}(\mathcal{Y}, \mathcal{X})$ of $\mathscr{E}$ is given by $\|\mathscr{G}\|^{-2} = 1 - \|\Gamma_{\mathscr{E}}\|^2$.

We call $\mathscr{D} \in \mathrm{TIC}(\mathcal{X}, \mathcal{Y})$ *inner* if $\mathscr{D}^*\mathscr{D} = I$ (on L^2, or equivalently, on L^2_+).

Theorem 5.10 (Inner) *Let $\mathscr{E} \in \mathrm{TI}(\mathcal{X}, \mathcal{Y})$ and $p \in [1, \infty]$. Then the claims (i)–(iii) are equivalent:*

- **(i)** *$\hat{\mathscr{E}}^*\hat{\mathscr{E}} = I$ (in $\mathrm{L}^\infty_{\mathrm{strong}}$);*
- **(i')** *$\mathscr{E}^*\mathscr{E} = I$ (on $\mathrm{L}^2(\mathcal{X})$);*
- **(i")** *$\|\mathscr{E} f\|_2 = \|f\|_2$ for every $f \in \mathrm{L}^2_+(\mathcal{X})$;*

(ii') $\langle \hat{\mathscr{E}}x', \hat{\mathscr{E}}x\rangle = \langle x', x\rangle$ *a.e. on* $\mathbb{R}$ *for every* $x, x' \in \mathcal{X}$*;*

(iii) $\|\hat{\mathscr{E}}x\|_{\mathcal{Y}} = \|x\|_{\mathcal{X}}$ *a.e. on* $\mathbb{R}$ *for every* $x \in \mathcal{X}$*.*

(Recall that further equivalent conditions are given in Theorem 4.10, by Lemma 5.1, with $F := \hat{\mathscr{E}}$, with $\mathbb{R}$ in place of $\mathbb{T}$ and with $\mathcal{P}$ replaced by $\tilde{\mathcal{P}}_2$, defined below Remark 6.2.)

Corollary 5.11 (Coprime) *Let* $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2$ *be Hilbert spaces. If a map* $\left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right] \in \mathrm{TIC}(\mathcal{X}, \mathcal{Y}_1 \times \mathcal{Y}_2)$ *is inner, then* $\mathscr{N}$ *and* $\mathscr{M}$ *are right coprime iff* $\|\Gamma_{\left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right]}\| < 1$.

Being *right coprime* means here that $\mathscr{P}\mathscr{M} + \mathscr{Q}\mathscr{N} = I$ for some $\mathscr{P}, \mathscr{Q} \in \mathrm{TIC}$, i.e., that $\hat{\mathscr{P}}\hat{\mathscr{M}} + \hat{\mathscr{Q}}\hat{\mathscr{N}} \equiv I$ on $\mathbb{C}^+$ for some $\hat{\mathscr{P}}, \hat{\mathscr{Q}} \in \mathcal{H}^\infty$.

An operator $\mathscr{D} \in \mathrm{TI}$ is uniquely determined by its Toeplitz operator $\pi_+\mathscr{D}\pi_+$ (or by $P_+\hat{\mathscr{D}}P_+$). Moreover, the following hold.

Proposition 5.12 (Causal, anti-causal and inner-outer)

Let $\mathscr{D} \in \mathcal{B}(\mathrm{L}^2(\mathcal{X}), \mathrm{L}^2(\mathcal{Y}))$.

Then $\mathscr{D} \in \mathrm{TIC}(\mathcal{X}, \mathcal{Y})$ *iff* $\pi_+\tau^t\mathscr{D}\pi_+ = \pi_+\mathscr{D}\tau^t\pi_+$ *for every* $t < 0$.

Let $\mathscr{D} \in \mathrm{TIC}(\mathcal{X}, \mathcal{Y})$.

Then $\mathscr{D}^{-1} \in \mathrm{TIC}(\mathcal{Y}, \mathcal{X})$ *iff* $\pi_+\mathscr{D}\pi_+$ *is invertible* $\mathrm{L}^2_+(\mathcal{X}) \to \mathrm{L}^2_+(\mathcal{Y})$.

Moreover, if $\mathscr{D}, \mathscr{D}^* \in \mathrm{TIC}$*, then* $\mathscr{D} \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$.

If $\mathscr{D} \in \mathrm{TIC}(\mathcal{X}, \mathcal{Y})$*,* $\mathscr{D}^*\mathscr{D} = I$ *and* $\overline{\mathscr{D}[\mathrm{L}^2_+(\mathcal{X})]} = \mathrm{L}^2_+(\mathcal{Y})$*, then* $\mathscr{D} = \mathscr{D}^{-*} \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$.

Theorem 5.13 (Divisor) *Assume that* $\mathscr{D} \in \mathrm{TIC}(\mathcal{X}, \mathcal{Y})$*,* $\mathscr{G} \in \mathrm{TIC}(\mathcal{Z}, \mathcal{Y})$ *for some Hilbert space* $\mathcal{Z}$*, and* $\mathscr{G}^*\mathscr{G} = I$*. Then* $\mathscr{D}[\mathrm{L}^2_+(\mathcal{X})] \subset \mathscr{G}[\mathrm{L}^2_+(\mathcal{Z})]$ *iff* $\mathscr{G}$ *is a left divisor of* $\mathscr{D}$*.*

The latter means that $\mathscr{D} = \mathscr{G}\mathscr{K}$ for some $\mathscr{K} \in \mathrm{TIC}(\mathcal{X}, \mathcal{Z})$. If $\mathscr{D}^*\mathscr{D} = I$, then $\mathscr{K}^*\mathscr{K} = I$.

We call $\mathcal{M} \subset \mathrm{L}^2_+(\mathcal{X})$ *translation-invariant* if $\tau^t\mathcal{M} = \mathcal{M}$ $(t < 0)$.

Theorem 5.14 (Lax–Halmos) *A closed subspace* $\mathcal{M}$ *of* $\mathrm{L}^2_+(\mathcal{X})$ *is translation-invariant iff* $\mathcal{M} = \mathscr{D}[\mathrm{L}^2_+(\mathcal{X}_0)]$ *for some closed subspace* $\mathcal{X}_0 \subset \mathcal{X}$ *and some inner* $\mathscr{D} \in \mathrm{TIC}(\mathcal{X}_0, \mathcal{X})$.

If also $\mathcal{M} = \mathscr{G}[\mathrm{L}^2_+(\mathcal{X}_1)]$ *for some Hilbert space* $\mathcal{X}_1$ *and some inner* $\mathscr{G} \in \mathrm{TIC}(\mathcal{X}_1, \mathcal{X})$*, then* $\mathscr{G} = \mathscr{D}T$ *for some* $T = T^{-*} \in \mathcal{B}(\mathcal{X}_1, \mathcal{X}_0)$.

We say that $\mathcal{M} \subset \mathrm{L}^2_+(\mathcal{X})$ *reduces translations* if $\pi_+\tau^t\mathcal{M} \subset \mathcal{M}$ $(t \in \mathbb{R})$.

Theorem 5.15 (Reducing subspace) *A closed subspace* $\mathcal{M}$ *of* $\mathrm{L}^2_+(\mathcal{X})$ *reduces translations iff* $\mathcal{M} = \mathrm{L}^2_+(\mathcal{X}_0)$ *for some closed subspace* $\mathcal{X}_0 \subset \mathcal{X}$*.*

A map $\mathscr{D} \in \mathrm{TIC}(\mathcal{X}, \mathcal{Y})$ is called *outer* if $\mathscr{D}[\mathrm{L}^2_+(\mathcal{X})]$ is dense in $\mathrm{L}^2_+(\mathcal{Y})$.

Theorem 5.16 (Inner-Outer Factorization) *Every $\mathscr{D} \in \mathrm{TIC}(\mathcal{X},\mathcal{Y})$ can be expressed as $\mathscr{D} = \mathscr{D}_i\mathscr{D}_o$, where $\mathscr{D}_o \in \mathrm{TIC}(\mathcal{X},\mathcal{Y}_0)$ is outer and $\mathscr{D}_i \in \mathrm{TIC}(\mathcal{Y}_0,\mathcal{Y})$ is inner, $\mathcal{Y}_0$ being a closed subspace of $\mathcal{Y}$. Moreover, $\|\mathscr{D}_o\|_{\mathrm{TIC}} = \|\mathscr{D}\|_{\mathrm{TIC}}$ and $\dim\mathcal{Y}_0 \le \dim\mathcal{X}$.*

If also $\mathscr{D} = \mathscr{D}_i'\mathscr{D}_o'$, where $\mathscr{D}_o' \in \mathrm{TIC}(\mathcal{X},\mathcal{Z}')$ is outer and $\mathscr{D}_i' \in \mathrm{TIC}(\mathcal{Z}',\mathcal{Y})$ is inner, $\mathcal{Z}'$ being a Hilbert space, then there exists $T = T^{-*} \in \mathcal{B}(\mathcal{Z}',\mathcal{Y}_0)$ such that $\mathscr{D}_i' = \mathscr{D}_iT$ and $\mathscr{D}_o' = T^*\mathscr{D}_o$.

If $\mathscr{D}$, $\mathscr{D}_o$ and $\mathscr{D}_i$ are as above, then $\mathscr{D}^*\mathscr{D} \ge \epsilon I$ for some $\epsilon > 0$ (i.e., $\mathscr{D}$ is left-invertible in TI) iff $\mathscr{D}_o$ is invertible in TIC. If it is, then $\mathscr{D}_o$ is called a (invertible) *spectral factor* of $\mathscr{D}^*\mathscr{D}$, because $\mathscr{D}_o^*\mathscr{D}_o = \mathscr{D}^*\mathscr{D}$.

Finally, we generalize Theorem 4.17. A function $F \in \mathcal{H}^2_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ is called *outer* if $M_F[\tilde{\mathcal{P}}_2(\mathcal{X})]$ is dense in $\mathcal{H}^2(\mathcal{Y})$. (This coincides with the above definition for $F \in \mathcal{H}^\infty$. The set $\tilde{\mathcal{P}}_2(\mathcal{X}) \subset \mathcal{H}^2(\mathcal{X})$ of certain rational functions is defined below Remark 6.2.)

Lemma 5.17 (Inner-Outer Factorization) *Every $F \in \mathcal{H}^2_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ can be expressed as $F = F_iF_o$, where $F_o \in \mathcal{H}^2_{\mathrm{strong}}(\mathcal{X},\mathcal{Y}_0)$ is outer and $F_i \in \mathcal{H}^\infty(\mathcal{Y}_0,\mathcal{Y})$ is inner, $\mathcal{Y}_0$ being a closed subspace of $\mathcal{Y}$. Moreover, $\|F_o\|_{\mathcal{H}^2_{\mathrm{strong}}} = \|F\|_{\mathcal{H}^2_{\mathrm{strong}}}$, $\|F_o\|_{\mathcal{H}^\infty} = \|F\|_{\mathcal{H}^\infty} \le \infty$, and $\dim\mathcal{Y}_0 \le \dim\mathcal{X}$.*

If also $F = F_i'F_o'$, where $F_o' \in \mathcal{H}^2_{\mathrm{strong}}(\mathcal{X},\mathcal{Z}')$ is outer and $F_i' \in \mathcal{H}^\infty(\mathcal{Z}',\mathcal{Y})$ is inner, $\mathcal{Z}'$ being a Hilbert space, then there exists $T = T^{-*} \in \mathcal{B}(\mathcal{Z}',\mathcal{Y}_0)$ such that $F_i' = F_iT$ and $F_o' = T^*F_o$.

Note that for still more general functions (in a weighted $\mathcal{H}^2_{\mathrm{strong}}$), the Cayley transform of Lemma 5.17 is contained in Remark 6.2.

6 Proofs for the real line

In this technical section we prove the results of Section 5.

The proofs in Section 4 could be rewritten for Section 5 except that on some results there are no separable versions in the literature for the real line, so the use of the Cayley Transform is the easiest way to prove these results.

In that setting, the shift S is mapped to the Laguerre shift S_{Lag} that maps f to $z \mapsto f(z)i(1-z)/(1+z)$ and $\mathcal{H}^2(\mathcal{Z})$ onto a weighted $\mathcal{H}^2$ space $\mathbb{C}^+ \to \mathcal{Z}$, for any Hilbert space $\mathcal{Z}$ [RR85].

Since in applications on $\mathbb{R}$ one usually wants to use the standard $\mathcal{H}^2(\mathbb{C}^+;\mathcal{Z})$ space instead of the weighted one and translations instead of the shift, we have also established the results given in Section 5, sometimes with non-straight-forward proofs, given in Lemma 6.3 below. The symbols B and B_2 stand for arbitrary Banach spaces.

All results on the unit circle can easily be converted for the real line (to their S_{Lag} form, some to the standard form too) by using the (extensions of the) well-known properties of the *Cayley Transform*

$$\phi : z \mapsto i\frac{1-z}{1+z} \qquad \text{and its inverse} \qquad \phi^{-1} : s \mapsto \frac{1-is}{1+is} \tag{52}$$

that are listed in Lemma 6.1 below. Here we sometimes write the domains and target spaces explicitly; e.g., $\mathrm{L}^p(\mathbb{T}; B)$ stands for L^p functions $\mathbb{T} \to B$; otherwise we refer to the "disc notation" of Sections 1–4.

Lemma 6.1 (Cayley Transform) *Let $1 \le p \le \infty$. The Cayley Transform ϕ maps $\mathbb{D} \to \mathbb{C}^+$ and $\mathbb{T} \to \mathbb{R} \cup \{\infty\}$ one-to-one and onto. Measurable (resp., null) sets (and only they) are mapped to measurable (resp., null) sets.*

The corresponding composite map $\cdot \circ \phi$ maps $\mathcal{H}^\infty(\mathbb{C}^+; B) \to \mathcal{H}^\infty(B)$,

$$\mathrm{L}^\infty(\mathbb{R}; B) \to \mathrm{L}^\infty(B), \quad \text{and} \quad \mathrm{L}^\infty_{\text{strong}}(\mathbb{R}; \mathcal{B}(B, B_2)) \to \mathrm{L}^\infty_{\text{strong}}(B, B_2) \tag{53}$$

isometrically onto. Measurable functions (and only they) are mapped to measurable functions.

The map $\diamondsuit_p f \mapsto \gamma_p \cdot (f \circ \phi)$ is an isometric isomorphism of $\mathrm{L}^p(\mathbb{R}; B)$ onto $\mathrm{L}^p(\mathbb{T}; B)$, where $\gamma_p(z) := (4\pi)^{1/p}/(1+z)^{2/p}$. Moreover, it maps $\mathcal{H}^p(\mathbb{C}^+; B)$ isometrically onto $\mathcal{H}^p(\mathbb{D}; B)$.

Therefore, $\heartsuit_p : T \mapsto \diamondsuit_p T \diamondsuit_p^{-1}$ maps

$$\mathcal{B}(\mathrm{L}^p(\mathbb{R}; B), \mathrm{L}^p(\mathbb{R}; B_2)) \quad \text{onto} \quad \mathcal{B}(\mathrm{L}^p(\mathbb{T}; B), \mathrm{L}^p(\mathbb{T}; B_2)) \tag{54}$$

isometrically. Moreover, for every $F \in \mathrm{L}^\infty_{\text{strong}}(\mathbb{R}; \mathcal{B}(B, B_2))$, we have $\heartsuit_p M_F = M_{F\circ\phi}$. Finally, $\heartsuit_p$ commutes with adjoints and valid compositions of operators and $\heartsuit_2$ with P_+ and P_-.

We set $\gamma := \gamma_2 = 2\sqrt{\pi}/(1+\cdot)$, $\diamondsuit := \diamondsuit_2$, $\heartsuit := \heartsuit_2$.

Proof: In the case where $p = 2$ and $B = \mathcal{X}$, $B_2 = \mathcal{Y}$, everything in the lemma can be found in Section 13.2 of [Mik02], particularly in Lemma 13.2.1 and in (b1), (b3) and (c1) of Theorem 13.2.3. Essentially the same proofs apply in the general case too; this is nonobvious only for the claims for $\diamondsuit_p$, for $p \neq 2$. Fortunately, those claims were given on pp. 128–131 on [Hof88], for general p in the scalar case; the same proofs apply in the general case too. (Note that in this report we have the additional constant $(2\pi)^{1/p}$ due to our normalized measure on $\mathbb{T}$.) $\square$

Thus, the inverse of $\heartsuit_p$ maps $\mathcal{H}^\infty_-$ (on $\mathbb{D}^-$) onto bounded holomorphic functions on $\mathbb{C}^- := \{s \in \mathbb{C} \mid \operatorname{Im} s < 0\}$, $\mathcal{H}^\infty$ on $\mathbb{D}$ onto bounded holomorphic functions on $\mathbb{C}^+$, and $\mathrm{L}^\infty_{\text{strong}}(B, B_2)$ onto $\mathrm{L}^\infty_{\text{strong}}(\mathbb{R}; \mathcal{B}(B, B_2))$, isometrically.

By simply Cayley Transforming all sets and operators in Section 4 we observe the following:

Remark 6.2 *All results in Section 4 hold also in their S_{Lag} forms.*

Note that here, e.g., the conditions (ii) and (iii) of Theorem 4.8 remain unchanged (except that $\mathcal{H}^\infty$ on $\mathbb{D}$ is mapped to $\mathcal{H}^\infty$ on $\mathbb{C}^+$) but, in (i'), $\mathcal{H}^2$ becomes its Cayley transform, a weighted $\mathcal{H}^2$ space on $\mathbb{C}^+$, and P_- in (i) undergoes an analogous change. (However, as shown in Lemma 6.3 below, also the original-looking conditions are equivalent to these; see Theorem 5.8.)

Note also that $\mathcal{P}(\mathcal{X})$ is thus replaced by $\tilde{\mathcal{P}}_p(\mathcal{X}) := \diamondsuit_p^{-1}\mathcal{P}(\mathcal{X}) \subset \mathcal{H}^p(\mathbb{C}^+;\mathcal{X})$. We have $\tilde{\mathcal{P}}_p(\mathbb{C})\mathcal{X} \subset \tilde{\mathcal{P}}_p(\mathcal{X}) \subset \mathcal{H}^p(\mathbb{C}^+;\mathcal{X})$ and $\tilde{\mathcal{P}}_p(\mathbb{C})\mathcal{X} \subset \mathcal{H}^p(\mathbb{C}^+;\mathbb{C})\mathcal{X}$. The functions in $\tilde{\mathcal{P}}(\mathcal{X}) := \tilde{\mathcal{P}}_2(\mathcal{X})$ are rational.

This extends the definition of "outer" to all functions of the form $\tilde{F} := F \circ \phi^{-1}$, where $F \in \mathcal{H}^2_{\text{strong}}(\mathbb{D};\mathcal{B}(\mathcal{X},\mathcal{Y}))$. Note that such functions contain all elements of $\mathcal{H}^2_{\text{strong}}(\mathbb{C}^+;\mathcal{B}(\mathcal{X},\mathcal{Y}))$. So we call such a function $\tilde{F}$ *outer* if F is outer. Thus, $\tilde{F}$ is outer iff $M_{\tilde{F}}[\tilde{\mathcal{P}}_2(\mathcal{X})]$ is a dense subset of $\mathcal{H}^2$ (and for the elements of $\mathcal{H}^\infty$ this coincides with the definition above Theorem 5.16). If $\tilde{F} \in \mathcal{H}^2_{\text{strong}}$, then $M_{\tilde{F}}[\tilde{\mathcal{P}}_2(\mathcal{X})] \subset \mathcal{H}^2$, but the converse is not true.

Note also that the formulation of Theorem 4.7(iv) becomes complex and useless under the Cayley Transform.

For such reasons, the results of Section 5 are more useful than those established in Remark 6.2 (although Corollary 5.5 and Theorem 5.6 coincide completely and others partially with the S_{Lag} forms).

Lemma 6.3 *The results in Section 5 hold. Moreover, Lemmata A.1 and A.2, Proposition A.6, Corollary A.7 and Theorem C.1 hold with the half-plane notation of Section 5 too (as well as in the S_{Lag} notation), mutatis mutandis (in particular, replace $\mathcal{P}(\mathcal{X})$ by $\tilde{\mathcal{P}}(\mathcal{X}) := \diamondsuit_p^{-1}[\mathcal{P}(\mathcal{X})]$).*

Proof: The results in Sections 2 and 3 (cf. Lemma 5.1) and in the appendices follow from the same proofs, mutatis mutandis, except that Theorem C.1 for $\mathbb{R}$ and $\mathbb{C}^+$ follows from the original Theorem (not its proof), because, by Lemma 6.1, we have $\|Fg\|_{\mathrm{L}^p(\mathbb{R};Y)} = \|\diamondsuit_p(Fg)\|_p = \|\heartsuit_p F \diamondsuit_p g\|_p$, $\|\heartsuit_p F\|_{\mathrm{L}^\infty_{\text{strong}}} = \|F\circ\phi\|_{\mathrm{L}^\infty_{\text{strong}}} = \|F\|_{\mathrm{L}^\infty_{\text{strong}}(\mathbb{R};\mathcal{B}(X,Y))}$ etc.

For the rest, the alternative S_{Lag} claims follow directly by Cayley Transforming the sets and operators (without using γ_p's). The standard forms follow from Lemma 6.1 except for the results concerning the shift; we shall treat them below.

1° *Proposition 5.12:* The first and the third claim are well known; see, e.g., [Wei91] and Lemma 2.1.7 of [Mik02]. For the others, the original proof will do, mutatis mutandis.

2° *Theorem 5.13:* The original proof will do.

3° *Theorem 5.14:* We only prove "only if", since the rest follows as in the original proof. As in the proof of the latter lemma on p. 106 of [Hof88], we observe that $\mathcal{M}$ is invariant under the multiplication by any $\mathcal{H}^\infty(\mathbb{C})$ function, hence so is $\diamondsuit[\mathcal{M}]$, hence $S[\diamondsuit\mathcal{M}] \subset \diamondsuit\mathcal{M}$, hence $\diamondsuit\mathcal{M} = M_{\tilde{F}}[\mathcal{H}^2(\mathcal{Y}_0)]$ for some closed subspace $\mathcal{Y}_0 \subset \mathcal{Y}$ and some inner $\tilde{F} \in \mathcal{H}^\infty(\mathcal{Y}_0,\mathcal{X})$, by Theorem 4.15.

But $\diamondsuit^{-1}M_{\tilde{F}}\diamondsuit = \heartsuit^{-1}M_{\tilde{F}} = M_F$, where $F := \tilde{F}\circ\phi^{-1}$, hence the function $F \in \mathcal{H}^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{Y}_0,\mathcal{X}))$ is inner and

$$\mathcal{M} = \diamondsuit^{-1}M_{\tilde{F}}[\mathcal{H}^2(\mathcal{Y}_0)] = M_F\diamondsuit^{-1}[\mathcal{H}^2(\mathcal{Y}_0)] = M_F[\mathcal{H}^2(\mathbb{C}^+;\mathcal{Y}_0)]. \tag{55}$$

4° *Theorem 5.15:* We only need to show "only if". By Theorem 5.14, there are $\mathcal{X}_0 \subset \mathcal{X}$ and $\mathscr{D} \in \mathrm{TIC}(\mathcal{X}_0,\mathcal{X})$ such that $\mathscr{D}[\mathrm{L}^2_+(\mathbb{C}^+;\mathcal{X}_0)] = \mathcal{M}$. Set

$$X' := \cup\{(\mathscr{D}fx)[\mathbb{R}_+] \,\big|\, f \in \mathcal{D},\ x \in \mathcal{X}_0\}, \qquad X := \overline{X'}, \tag{56}$$

where $\mathcal{D}$ is the set of compactly supported $\mathcal{C}^\infty$ functions $\mathbb{R}_+ \to \mathbb{C}$; by $\mathscr{D}fx$ we refer to the continuous [Sta05, Theorem 2.6.6 & Corollary 4.6.13] representative of $[\mathscr{D}fx] \in \mathrm{L}^2(\mathbb{R}_+; \mathcal{X}_0)$.

Obviously, $\mathscr{D}fx \in \mathrm{L}^2(\mathbb{R}_+; X)$ for all $f \in \mathcal{D}$, $x \in \mathcal{X}_0$, hence $\mathcal{M} \subset \mathrm{L}^2(\mathbb{R}_+; X)$, because the linear combinations of functions of the form fx are dense in $\mathrm{L}^2(\mathbb{R}_+; \mathcal{X}_0)$ [Mik02, Theorem B.3.11(b1)]. Thus, we only need to show that $\mathrm{L}^2(\mathbb{R}_+; X) \subset \mathcal{M}$, which will be done below.

4.1° *For any $\tilde{x} \in X$ and $\epsilon > 0$, there exist $\delta > 0$ and $g \in \mathcal{M}$ such that $\|\tilde{x} - g(t)\| < \epsilon$ on $[0, \delta)$ and $g \equiv 0$ elsewhere:* By the definition of X, there exist $f \in \mathcal{D}$, $x \in X_0$ and $r > 0$ such that $\|\tilde{x} - (\mathscr{D}fx)(r)\| < \epsilon$. Pick $\delta > 0$ such that $\|\tilde{x} - (T_F fx)(r+t)\| < \epsilon$ for $0 \le t < \delta$. Then $g := \pi_{[\delta,\infty)}\tau^r \mathscr{D}fx$ satisfies the requirements, where $\pi_{[\delta,\infty)} := I - (\pi_+)\tau^{-\delta}\pi_+\tau^\delta$.

4.2° *We have $\mathrm{L}^2(\mathbb{R}_+; X) \subset \mathcal{M}$:* This follows from 4.1°, because 1. any step function can be estimated (in $\mathrm{L}^2(\mathbb{R}_+; X)$) by a finite sum of translations of the functions of the form $\tilde{x}\chi_{[0,\delta)}$, where $\tilde{x} \in X$ and χ_E is the characteristic funtion of E; 2. any finite-dimensional function can be estimated by a step function (since the Lebesgue measure is outer regular and an open set is a countable union of intervals); and 3. any $\mathrm{L}^2(\mathbb{R}_+; X)$ function can be estimated by a finite-dimensional function.

5° *Lemma 5.17:* For any F such that $\heartsuit F \in \mathcal{H}^2_{\text{strong}}(\mathcal{X}, \mathcal{Y})$, we get $\heartsuit F = (\heartsuit F_i)(\heartsuit F_o)$ from Theorem 4.17. The function $F_i \in \mathcal{H}^\infty(\mathbb{C}^+; \mathcal{B}(\mathcal{X}_0, \mathcal{Y}))$ is inner and $\heartsuit F_o$ is outer, hence so is F_o (see below this proof). Here $\heartsuit^{-1}[\mathcal{H}^2_{\text{strong}}]$ becomes the weighted $\mathcal{H}^2_{\text{strong}}$ on $\mathbb{C}^+$ (hence it contains $\mathcal{H}^2_{\text{strong}}(\mathbb{C}^+; \mathcal{B}(\cdot,\cdot))$).

However, the theorem also holds for $\mathcal{H}^2_{\text{strong}}(\mathbb{C}^+; \mathcal{B}(\cdot,\cdot))$, not merely for its weighted variant. Indeed, since F_i is inner, it preserves the $\mathcal{H}^2$ norm, hence, $\|F_o x\|_{\mathcal{H}^2} = \|Fx\|_{\mathcal{H}^2}$ for each $x \in \mathcal{X}$, hence $\|F_o\|_{\mathcal{H}^2_{\text{strong}}} = \|F\|_{\mathcal{H}^2_{\text{strong}}} \le \infty$. □

7 Real Hilbert spaces

In this section we note that our methods also work on real Hilbert spaces. We use the following fact.

Lemma 7.1 *If W is an orthonormal basis of a real Hilbert space $\mathcal{Z}$, then $\mathcal{Z} + i\mathcal{Z}$ with natural operations is a complex Hilbert space, and W is an orthonormal basis of $\mathcal{Z} + i\mathcal{Z}$.*

(The simple proof is left to the reader. Here $i(x + ix') := -x' + ix$. The inner product is given by $(x+ix', v+iv') := (x, v)+(x', v')+i(x', v)-i(x, v')$.)

Theorem 7.2 *Proposition 2.2, Lemmata A.1, A.2, A.3, A.8 and A.10 and the results of Sections 3 and B (omitting those for the $\mathcal{H}^p$ and $\mathcal{H}^p_{\text{strong}}$ classes) also hold under the alternative assumption that all Hilbert spaces are real.*

Proof: Most claims follow from the same proofs. For, e.g., Proposition 2.2, we can also extend $T \in \mathcal{B}(\mathcal{X}, \mathrm{L}^\infty(\mathcal{Y}))$ to $(x + ix') \mapsto Tx + iT'x$ and then take the real part of the corresponding F. □

Some results, such as Theorems 4.14 and 4.16, readily translate to the case of real Hilbert spaces. However, most others require much more work.

When one wants to establish the remaining results of Sections 3–6 to this case, the original separable-case proofs usually do not directly provide "real" results, and there are some problems with holomorphicity and with Fourier and Z-transforms, which require complex scalars. These problems can be overcome by first extending operators and functions by replacing $\mathcal{X}$ by its complexification (the complex Hilbert space) $\mathcal{X}+i\mathcal{X}$ and $\mathcal{Y}$ by $\mathcal{Y}+i\mathcal{Y}$. However, then one must rewrite the original (separable case) proofs to show that the resulting spaces and functions can be taken "real" to obtain the final results for the original real Hilbert spaces (not for their complexifications).

Most of this work is done in [Mik06b], which thus presents the real-Hilbert-space variants of most results in this report but also many other related results, including more control-theoretic ones, and further references on real Hilbert spaces.

By "real" functions we mean real-symmetric ones in the sense of Lemma 7.3 and Theorem 7.4 below (they use the half-plane notation of Section 5). Indeed, "real-valued" (i.e., essentially $\mathcal{Y}$-valued) functions are the ones whose transforms are real-symmetric:

Lemma 7.3 *Let $f \in \mathrm{L}^2(\mathbb{R};\mathcal{Y}+i\mathcal{Y})$. Then $f \in \mathrm{L}^2(\mathbb{R};\mathcal{Y})$ iff $\langle \widehat{f}(-r), y\rangle = \overline{\langle \widehat{f}(r), y\rangle}$ a.e. for every $y \in \mathcal{Y}$.*

Proof: Write $\Lambda w := \langle w, y\rangle$.

1° If $f \in \mathrm{L}^2(\mathbb{R};\mathcal{Y})$, then $\Lambda\widehat{f}(-r) = \int_{\mathbb{R}} \mathrm{e}^{irt}\Lambda f(t)\,dt = \overline{\int_{\mathbb{R}} \mathrm{e}^{-irt}\Lambda f(t)\,dt} = \overline{\widehat{\Lambda f}(r)}$.

2° Conversely, let $f = f_R + i f_I$, $f_R, f_I \in \mathrm{L}^2(\mathbb{R};\mathcal{Y})$ be such that $\langle \widehat{f}(-r), y\rangle = \overline{\langle \widehat{f}(r), y\rangle}$ a.e. But $\langle \widehat{f_R}(-r), y\rangle = \overline{\langle \widehat{f_R}(r), y\rangle}$ a.e., by 1°, hence

$$i\overline{\langle \widehat{f_I}(r), y\rangle} = \langle i\widehat{f_I}(-r), y\rangle = \overline{\langle i\widehat{f_I}(r), y\rangle} = -i\overline{\langle \widehat{f_I}(r), y\rangle} \quad \text{a.e.} \tag{57}$$

This holds for every $y \in \mathcal{Y}$, hence $\widehat{f_I} = 0$ a.e. □

Analogously, "real" TI maps are those whose symbols are real-symmetric:

Theorem 7.4 *Any $\mathscr{E} \in \mathrm{TI}(\mathcal{X},\mathcal{Y})$ can be naturally extended to $\mathscr{E} \in \mathrm{TI}(\mathcal{X}+i\mathcal{X},\mathcal{Y}+i\mathcal{Y})$ by $\mathscr{E}(f+ig) := \mathscr{E}f + i\mathscr{E}g$.*

The operators in $\mathrm{TI}(\mathcal{X}+i\mathcal{X},\mathcal{Y}+i\mathcal{Y})$ of this form are exactly those for which $\hat{\mathscr{E}}$ is real-symmetric*, i.e.,*

$$\langle \hat{\mathscr{E}}x, y\rangle(-r) = \overline{\langle \hat{\mathscr{E}}x, y\rangle(r)} \text{ for a.e. } r \in \mathbb{R} \tag{58}$$

for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$.

(Note that in the nonseparable case "a.e." may depend on the vector x.)
Proof: Let $f \in \mathrm{L}^2(\mathbb{R};\mathbb{R})$. Set $F := \langle \mathscr{E} f x, y\rangle \in \mathrm{L}^2(\mathbb{R};\mathbb{R})$. By Theorem 5.2, we have $\hat{\mathscr{E}} \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{X}+i\mathcal{X},\mathcal{Y}+i\mathcal{Y})$, and $\langle \hat{\mathscr{E}} \widehat{f} x, y\rangle = \langle \mathcal{F}(\mathscr{E} f x), y\rangle = \widehat{F}$. Thus, (58) follows from Lemma 7.3.

Conversely, if $\hat{\mathscr{E}}$ is real-symmetric, then, as above, we see that $g := \mathscr{E} f x$ satisfies $\langle \widehat{g}(-r), y\rangle = \overline{\langle \widehat{g}(r), y\rangle}$ a.e. for every $y \in \mathcal{Y}$, so then $g \in \mathrm{L}^2(\mathbb{R};\mathcal{Y})$, by Lemma 7.3. Because the closed span of such functions fx equals $\mathrm{L}^2(\mathbb{R};\mathcal{Y})$ (recall that simple functions are dense), we have $\mathscr{E} f \in \mathrm{L}^2(\mathbb{R};\mathcal{Y})$ for every $f \in \mathrm{L}^2(\mathbb{R};\mathcal{Y}$, i.e., $\mathscr{E} \in \mathrm{TI}(\mathcal{X},\mathcal{Y})$. □

8 Notes

The contents of Section 2 are mostly from [Mik08] and [Mik02, Appendix F] (or older in the separable case). Section 3 seems to be new.

The results in Section 4 seem to be new in the nonseparable case except Theorem 4.3, Lemma 4.12 and Proposition 4.13, as explained in their proofs. However, probably none of those results is new in the separable case. Most of them can be found in [Nik02], [Pel03], [Nik86], [RR85] or in other similar monographs, as explained in Section 4 and below. These monographs also record the history of the results.

The operator-valued versions of Theorems 4.4 and 4.7 and Corollary 4.5 are due to [Pag70]. Historical notes on those results are given on p. 84 of [Pel03]. The operator-valued version of Theorem 4.6 is due to [Tre85].

The operator-valued version of Theorem 4.8 was established in [Tre04] but the equivalence of (i) and (ii) was given already in [Arv75] and [SF76]. The estimates for $\tilde{F}$ and $\begin{bmatrix} F & \tilde{F} \end{bmatrix}^{-1}$ in Theorem 4.8 are due to Sergei Treil. The latter estimate can be improved.

Theorem 4.9 is essentially given on p. 203 of [Nik86] in the scalar case. The equivalence (and (b)) in Theorem 4.10 is essentially well known in the separable case. Corollary 4.11 has been established at least in [CO06] (in the separable case), with a constructive proof.

Our proof of Theorem 4.14 is from p. 240 of [FF90]. Theorem 4.16 was given in [RR85]. The history of the Beurling–Lax–Halmos Theorem 4.15 (resp., inner-outer factorization 4.17) is explained on p. 21 (resp., 107–108) of [RR85]. The shift-invariant subspaces of $\mathrm{L}^2(\mu)$ can be found in [Nik86, pp. 14–17] (the separable case). For the Nevanlinna class N the inner-outer factorization was given on p. 100 of [RR85] (note that their "inner" allows also partial isometries and that $\mathrm{N} \not\subset \mathcal{H}^2_{\text{strong}}$ and $\mathcal{H}^2_{\text{strong}} \not\subset \mathrm{N}$), but our version is from [Nik86] and [FF90].

The author did not find in the literature even a scalar version of Theorem 5.15 nor any infinite-dimensional versions of Theorems 5.14 and 5.16. Finite-dimensional and scalar versions of Theorems 5.14 and 5.16, respectively, are given in [Lax59].

Lemmata A.3–A.5 and Proposition A.6 are known at least in some generality. The statement of Lemma A.10 is due to Sergei Treil.

Related results for (possibly) nonseparable Hilbert spaces are given in Chapters 1–3 of [RR85], in [Mik08], and in [Mik02], particularly in Sections 13.1, 6.4, 6.5, Chapters 2–5, and in Appendix F.

A Auxiliary results

In this appendix we list several, often well-known results on Hilbert spaces and on vector- or operator-valued functions.

By $\dim \mathcal{X}$ we denote the cardinality of an arbitrary orthonormal basis of $\mathcal{X}$ (it is independent of the basis [Mik02, Lemma A.3.1(a1)]). Thus, "$\dim \mathcal{X} \leq \dim \mathcal{Y}$" means that there exists a one-to-one map of an orthonormal basis of $\mathcal{X}$ into an orthonormal basis of $\mathcal{Y}$ (such a map is a linear isometry). We need the following facts.

Lemma A.1 (dim) **(a)** *If* $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, *then* $\dim \mathcal{X} \geq \dim \overline{T[\mathcal{X}]}$.

(b) *If* $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ *and* $T^*T \geq \epsilon I$, *then* $\dim \mathcal{X} = \dim T[\mathcal{X}] \leq \dim \mathcal{Y}$.

(c) *If* $\mathcal{X}$ *is infinite-dimensional, then* $\dim \mathrm{L}^2(\mathcal{X}) = \dim \mathcal{H}^2(\mathcal{X}) = \dim \mathcal{X}$.

Proof: (For extensions and further details see Theorem A.3.1(a3)&(a4) and Theorem B.1.16 of [Mik02].)

(a) Assume that $\dim \mathcal{X} = \infty$ (otherwise (a) is obvious). Let Q be a dense, countable subset of $\mathbb{C}$. Let X and Y be orthonormal bases of $\mathcal{X}$ and $\mathcal{Y}_T := \overline{T[\mathcal{X}]}$, respectively. Then $S := \operatorname{span}\{qx \,\big|\, q \in Q,\ x \in X\}$ is dense in $\mathcal{X}$, hence $T[S]$ is dense in $\mathcal{Y}_T$. Thus, for any $y \in Y$ there exists a $s_y \in S$ such that $\|y - Ts_y\| < 1/2$. Since $y \neq y' \ \Rightarrow\ s_y \neq s_{y'}$, we have

$$\dim \mathcal{Y}_T = \operatorname{card} Y \leq \operatorname{card} T[S] = \operatorname{card} S = \operatorname{card} X = \dim \mathcal{X}. \tag{59}$$

(b) Now $\mathcal{Y}_T := T[\mathcal{X}]$ is closed and also $T^* : \mathcal{Y}_T \to \mathcal{X}$ is onto, so $\dim \mathcal{Y}_T \geq \dim T^*[\mathcal{Y}_T] = \dim \mathcal{X}$, by (a). By (a), $\dim \mathcal{X} \leq \dim \mathcal{Y}_T$, so (b) holds.

(c) Given orthonormal bases $\mathcal{U}$ of $\mathcal{X}$ and $\mathcal{F}$ of $\mathrm{L}^2(\mathbb{C})$ (resp., of $\mathcal{H}^2(\mathbb{C})$), the set $\{fx \,\big|\, f \in \mathcal{F},\ x \in \mathcal{U}\}$ is an orthonormal basis of $\mathrm{L}^2(\mathcal{X})$ (resp., of $\mathcal{H}^2(\mathcal{X})$), of the cardinality of $\mathcal{U} \times \mathcal{F} \approx \mathcal{U}$ (because $\mathcal{F} \approx \mathbb{N}$). $\square$

Note that if $\dim \mathcal{X} = \infty$, then $\dim \mathcal{X} = \operatorname{card} \mathbb{N}$ iff $\mathcal{X}$ is separable (i.e., iff $\mathcal{X}$ is isomorphic to $\ell^2(\mathbb{N})$).

In the case of an inner function, the output space cannot have a smaller dimension than the input space (here $M_F : \mathcal{H}^2(\mathcal{X}) \to \mathcal{H}^2(\mathcal{Y})$ refers to the operator $f \mapsto Ff$):

Lemma A.2 (dim) *Assume that* $F \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{X}, \mathcal{Y})$ *is satisfies* $M_F^* M_F \geq \epsilon I$ *for some* $\epsilon > 0$. *Then* $\dim \mathcal{X} \leq \dim \mathcal{Y}$ *and hence* $\mathcal{X}$ *is isometrically isomorphic to a closed subspace, say* $\tilde{\mathcal{Y}}$, *of* $\mathcal{Y}$, *i.e.,* $T\mathcal{X} = \tilde{\mathcal{Y}}$ *for some* $T = T^{-*} \in \mathcal{B}(\mathcal{X}, \tilde{\mathcal{Y}})$.

Proof: 1° It is well-known that if $\mathcal{X}$ is separable, then $F(z)^*F(z) \geq \epsilon I$ for a.e. $z \in \mathbb{T}$, so then $\dim \mathcal{X} \leq \dim \mathcal{Y}$, by Lemma A.1(b).

2° Assume that $\mathcal{X}$ is nonseparable. By Lemma A.1(c)&(b), we have $\dim \mathcal{X} = \dim \mathcal{H}^2(\mathcal{X}) \leq \dim \mathcal{H}^2(\mathcal{Y})$. Therefore, $\mathcal{H}^2(\mathcal{Y})$ is nonseparable, hence so is $\mathcal{Y}$. Consequently, $\dim \mathcal{Y} = \dim \mathcal{H}^2(\mathcal{Y}) \geq \dim \mathcal{X}$. □

Sometimes we need to build a Hilbert space as the direct sum of a collection of (not necessarily disjoint) Hilbert spaces. The following is obvious:

Lemma A.3 (Direct sum) *If Z_X is a Hilbert space for each $X \in \mathcal{Q}$, and we set $\|z\|_{\mathcal{Z}}^2 := \sum_{X \in \mathcal{Q}} \|z(X)\|_{Z_X}^2$, then $\mathcal{Z} := \{z \in \prod_{X \in \mathcal{Q}} Z_X \mid \|z\|_{\mathcal{Z}} < \infty\}$ becomes a Hilbert space when equipped with the inner product $\langle z, w \rangle_{\mathcal{Z}} := \sum_{X \in \mathcal{Q}} \langle z(X), w(X) \rangle_{Z_X}$.*

Linear combinations of Fourier coefficients converge to an L^p function:

Lemma A.4 (Fejér) *Let X and Y be Banach spaces, $1 \leq p < \infty$ and $f \in \mathrm{L}^p(X)$. Set*

$$\widehat{f}(k) := \frac{1}{2\pi} \int_{\mathbb{T}} z^{-k} f(z)\, dz \qquad (k \in \mathbb{Z}). \tag{60}$$

Then $K_n * f \to f$ in $\mathrm{L}^p(X)$, hence a subsequence converges a.e., as $n \to \infty$, where

$$K_n * f := \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) z^k \widehat{f}(k). \tag{61}$$

If $f \in \mathcal{C}(X)$, then $K_n * f \to f$ uniformly, as $n \to \infty$.

If $G \in \mathrm{L}^\infty(\mathcal{B}(X,Y))$, then $\int_{\mathbb{T}} (K_n * G)(z) g(z)\, dz \to \int_{\mathbb{T}} G(z) g(z)\, dz$ for all $g \in \mathrm{L}^1(X)$.

Proof: By the properties of the Fejér's kernel $\{K_n\}$ ([Kat76], pp. 9–12), we have uniform convergence (hence in L^p too) for continuous f. Since the convolution is continuous $\mathrm{L}^1 \times \mathrm{L}^p \to \mathrm{L}^p$, by the Minkovski Integral Inequality (Theorem B.4.16(b) of [Mik02]), and continuous functions are dense in L^p, we get the general case.

The last claim follows from the Hölder Inequality, because the above also holds for $\{K_n(-\cdot)\}$, and $\int_{\mathbb{T}} (K_n * G)(z) g(z)\, dz = \int_{\mathbb{T}} G(s)(K_n(-\cdot) * g)(s)\, ds$, by the Fubini Theorem. □

Using a strong integral, we can compute the Fourier coefficients of any $\mathrm{L}^1_{\text{strong}}$ function $F : \mathbb{T} \to \mathcal{B}(X,Y)$:

Lemma A.5 ($\widehat{\mathrm{L}^p_{\text{strong}}} \subset \ell^\infty$) *Let X and Y be Banach spaces, $1 \leq p \leq \infty$, and $F \in \mathrm{L}^p_{\text{strong}}(X,Y)$. Then $\widehat{F} : \mathbb{Z} \to \mathcal{B}(X,Y)$ satisfies $\|\widehat{F}\|_\infty \leq \|F\|_{\mathrm{L}^p_{\text{strong}}}$ where $\widehat{F}(k) := \frac{1}{2\pi} \int_{\mathbb{T}} z^{-k} F(z)\, dz$ for each $k \in \mathbb{Z}$.*

Moreover, for each $x \in X$, we have

$$(Fx)(z) = \lim_{n\to\infty} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) z^k \widehat{F}(k)x \tag{62}$$

in $\mathrm{L}^p(Y)$*, hence a subsequence converges for a.e.* $z \in \mathbb{T}$*.*

Proof: By the Hölder Inequality, $\|F\|_{\mathrm{L}^1_{\text{strong}}} \le \|F\|_{\mathrm{L}^p_{\text{strong}}}$. Obviously, the function $\widehat{F}(k) : x \mapsto \int_{\mathbb{T}} z^{-k} F(z)x\,dz$ is linear and

$$\|\widehat{F}(k)x\|_Y \le \|Fx\|_{\mathrm{L}^1(Y)} \le \|F\|_{\mathrm{L}^1_{\text{strong}}} \|x\|_X, \tag{63}$$

so the integral converges in the strong sense (not necessarily as a Bochner integral) and its value satisfies $\|\widehat{F}\|_\infty \le \|F\|_{\mathrm{L}^1_{\text{strong}}}$. The last claims follow from Lemma A.4. □

Functions are holomorphic on $\mathbb{D}$ iff their negative Fourier coefficients are zero:

Proposition A.6 *Let* $F \in \mathrm{L}^p_{\text{strong}}(X,Y)$ *(resp.,* $\mathrm{L}^p(X)$*), where* X *and* Y *are Banach spaces and* $1 \le p \le \infty$*.*

Then $F \in \mathcal{H}^p_{\text{strong}}(X,Y)$ *(resp.,* $\mathcal{H}^p(X)$*) iff*

$$\widehat{F}(n) = 0 \quad \textit{for} \quad n = -1, -2, -3, \ldots, \tag{64}$$

or equivalently, iff the Poisson integral (see (6)) of F *is holomorphic* $\mathbb{D} \to \mathcal{B}(X,Y)$ *(resp.,* $\mathbb{D} \to X$*).*

By $\mathrm{L}^p \cap \mathcal{H}^p$ we mean functions $F \in \mathrm{L}^p$ whose Poisson integral is in $\mathcal{H}^p$, or equivalently, functions $F \in \mathcal{H}^p$ that have a radial (equivalently, nontangential) limit (a.e.) function in L^p (similarly for $\mathrm{L}^p_{\text{strong}} \cap \mathcal{H}^p_{\text{strong}}$; cf. Theorem 3.3.1(e)&(a1) of [Mik02]). Note that not all elements of $\mathcal{H}^p(X)$ are of this form for general (non-Hilbert) X.

Proof of Proposition A.6: We prove the L^p case; the other case follows. The scalar case is well known (see, e.g., Theorem 4.7C, p. 89 of [RR85]) and leads to "only if". Conversely, if $\widehat{F}(n) = 0$ for $n = -1, -2, -3, \ldots$, then the Poisson integral of ΛF is holomorphic for each $\Lambda \in X^*$, by the scalar case, hence F is holomorphic on $\mathbb{D}$. By the properties of the Poisson integral (e.g., Lemma D.1.8(d)&(a3)) of [Mik02]), we have $F \in \mathcal{H}^p$. □

The following says, among other things, that finite-dimensional $\mathcal{H}^p$ functions are dense in $\mathcal{H}^p(\mathcal{X})$ (when $\mathcal{X}$ is a Hilbert space):

Corollary A.7 *If* $1 \le p < \infty$ *and* $\mathcal{E}$ *is a collection of closed subspaces of* $\mathcal{X}$*, then the set of finite linear combinations of the spaces* $\mathcal{H}^p(X)$ *(*$X \in \mathcal{E}$*) is dense in* $\mathcal{H}^p(\overline{\operatorname{span} \cup \mathcal{E}})$*.*

Proof: Given $f \in \mathcal{H}^p(X)$ and $\epsilon > 0$, we have $f \in \mathrm{L}^p(X)$, Proposition 2.3. By Lemma A.4, $g := K_n * f$ satisfies $\|g - f\|_p < \epsilon$ for n big enough. But g is of the form $\sum_{k=0}^n z^k x_k$, by Proposition A.6, and each $z^k x_k$ can be estimated (in $\mathcal{H}^p$) by a linear combination of elements of $\cup_{X \in \mathcal{E}}\{z^k x \mid x \in X\}$. □

The $\mathrm{L}^\infty_{\text{strong}}$ and L^∞ norms coincide for Bochner-measurable functions:

Lemma A.8 ($\mathrm{L}^\infty \subset \mathrm{L}^\infty_{\text{strong}}$) *Let X and Y be Banach spaces and let $(Q, \mathfrak{M}, \mu)$ be a complete positive measure space. For any Bochner-measurable $F : Q \to \mathcal{B}(X,Y)$ we have $\|F\|_\infty = \|F\|_{\mathrm{L}^\infty_{\text{strong}}}$; in particular, $F \in \mathrm{L}^\infty \Leftrightarrow F \in \mathrm{L}^\infty_{\text{strong}}$.*

Thus, $\mathrm{L}^\infty(Q; \mathcal{B}(X,Y))$ is a closed subspace of $\mathrm{L}^\infty_{\text{strong}}(Q; \mathcal{B}(X,Y))$.

If X is separable, then $\|[F]\|_{\mathrm{L}^\infty_{\text{strong}}} = \operatorname{ess\,sup} \|F\|_{\mathcal{B}(X,Y)}$ for every $F \in \mathrm{L}^\infty_{\text{strong}}$.

In the setting of Example C.2, the last claim above does not hold and each equivalence class contains also non-measurable elements.

Proof: 1° If $F : Q \to \mathcal{B}(X,Y)$ is countably-valued and measurable, i.e., $F = \sum_{k=0}^\infty T_k \chi_{E_k}$, where $T_k \in \mathcal{B}(X,Y)$ and $E_k \in \mathfrak{M}$ for each k, the sets E_k being disjoint, then

$$\|F\|_{\mathrm{L}^\infty_{\text{strong}}} = \sup_{\mu(E_k) > 0} \|T_k\| = \|F\|_\infty \leq \infty. \tag{65}$$

2° For a general Bochner-measurable $F : Q \to \mathcal{B}(X,Y)$ we have $\|F_n - F\|_\infty \to 0$, as $n \to \infty$, where functions $\{F_n\}$ are countably-valued and measurable, by Corollary 1 on p. 73 of [HP57]. Obviously, $\|F_n - F\|_{\mathrm{L}^\infty_{\text{strong}}} \leq \|F_n - F\|_{\mathrm{L}^\infty} \to 0$, hence $\|F\|_{\mathrm{L}^\infty_{\text{strong}}} = \lim_n \|F_n\|_{\mathrm{L}^\infty_{\text{strong}}} = \lim_n \|F_n\|_\infty = \|F\|_\infty \leq \infty$.

3° Let $\{x_k\}_{k=0}^\infty$ be dense in the unit ball of X. If $M < \operatorname{ess\,sup} \|F\|$, then $\|F\|_{\mathcal{B}} > M$ on E, where $\mu(E) > 0$. But $E = \cup_k E_k$, where

$$E_k := \{q \in E \mid \|F(q)x_k\| > M\}, \tag{66}$$

so $\mu(E_n) > 0$ for some n and hence $\|F\|_{\mathrm{L}^\infty_{\text{strong}}} = \sup_k \|Fx_k\|_{\mathrm{L}^\infty} \geq \|Fx_n\|_{\mathrm{L}^\infty} > M$. Since $M < \operatorname{ess\,sup} \|F\|$ was arbitrary, we have $\|F\|_{\mathrm{L}^\infty_{\text{strong}}} \geq \operatorname{ess\,sup} \|F\|$. The converse is obvious. □

Lemma A.9 *If $f : Q \to B$ is Bochner-measurable, where Q is a measure space and B is a Banach space, then $\|f\|_\infty = \sup_{\Lambda \in B^*,\ \|\Lambda\| \leq 1} \|\Lambda f\|_\infty$.*

Proof: Since f is almost separably-valued, we can assume that B is separable. Pick $\Lambda_1, \Lambda_2, \ldots \in B^*$ such that $\|\Lambda_k\| \leq 1$ and $\|x\| = \sup_k |\Lambda_k x|$ for every $x \in B$.

If $M := \sup_k \|\Lambda_k f\|_\infty < \infty$, then $|\Lambda_k f| \leq M$ a.e. for every k, hence then $\|f\|_B \leq M$ a.e.; consequently, then $\|f\|_\infty \leq M$. Then converse is obvious. □

In a Hilbert space, a projection has the same norm as its complementary projection:

Lemma A.10 *If $P = P^2 \in \mathcal{B}(\mathcal{X})$ and $0 \neq P \neq I$, then $\|P\| = \|I - P\| \geq 1$.*

Proof: Set $Q := I - P$, $U_P := \{Px \mid \|Px\| = 1\}$, $U_Q := \{Qx \mid \|Qx\| = 1\}$. Let Q' denote the orthogonal projection $\mathcal{X} \to Q[\mathcal{X}]$, $Q^\perp := 1 - Q'$. for each $x \in \mathcal{X}$. Since $\|P\| = \sup_{x \neq 0} \|Px\|/\|x\| = \sup_{\|Px\|=1} \|x\|^{-1}$, we have

$$\begin{aligned}
\|P\|^{-2} &= \inf_{\|Px\|=1} \|x\|^2 &&= \inf_{\|Px\|=1} \|Px + Qx\|^2 && (67)\\
&= \inf_{p \in U_P,\ q \in Q[\mathcal{X}]} \|p + q\|^2 &&= \inf_{p \in U_P,\ q \in Q[\mathcal{X}]} \|Q^\perp p + Q'p + q\|^2 && (68)\\
&= \inf_{p \in U_P} \|Q^\perp p\|^2 &&= \inf_{p \in U_P} (1 - \|Q'p\|^2) && (69)\\
&= \inf_{p \in U_P} (1 - \sup_{q \in U_Q} |\langle p, q\rangle|^2) &&= 1 - \inf_{p \in U_P,\ q \in U_Q} |\langle p, q\rangle|^2 && (70)
\end{aligned}$$

$= \|Q\|^{-2}$ (exchange the roles of P and Q for this last equality). □

B Postponed proofs for Section 3

In this section we prove Theorem 3.2 and Lemma 3.3.

We start with an auxiliary result (Lemma B.1). It says that if, e.g., we study the effects of $F : \mathbb{D} \to \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $G : \mathbb{D} \to \mathcal{B}(\mathcal{Y}, \mathcal{X})$ on separable sets $X_0 \subset \mathcal{X}$ and $Y_0 \subset \mathcal{Y}$, we can without loss of generality assume that $\mathcal{X}$ and $\mathcal{Y}$ are separable.

Moreover, if $\mathcal{E} \in \mathcal{B}(\mathrm{L}^2(\mathcal{X}), \mathrm{L}^2(\mathcal{Y}))$, and $X_0 \subset \mathcal{X}$ is separable, then we have $\mathcal{E}[\mathrm{L}^2(X_0)] \subset \mathrm{L}^2(\tilde{Y})$ for some closed, separable subspace $\tilde{Y} \subset \mathcal{Y}$. These and similar facts will be given in the lemma below (To obtain the result mentioned above, take $F := \mathcal{E}$, $G := 0$, $A := B := \mathrm{L}^2$ with $\mathcal{J}$ (resp., $\mathcal{K}$) being the collection of all closed vector subspaces of $\mathcal{X}$ (resp., $\mathcal{Y}$).)

Lemma B.1 *Let $\mathcal{J}$ and $\mathcal{K}$ be collections of topological spaces. Assume that for each $X \in \mathcal{J}$ the set $A(X)$ is a topological vector space and the following conditions hold:*
1. If X is separable, then so is $A(X)$.
2. If $\tilde{X} \in \mathcal{J}$ and $\tilde{X} \subset X$, then $A(\tilde{X})$ is a closed subspace of $A(X)$.
3. If $X' \subset X$ is separable, then $X' \subset \tilde{X}$ for some closed, separable $\tilde{X} \subset X$ satisfying $\tilde{X} \in \mathcal{J}$.
4. If $f \in A(X)$, then $f \in A(\tilde{X})$ for some closed, separable $\tilde{X} \subset X$ satisfying $\tilde{X} \in \mathcal{J}$.
5. If $E \subset \mathcal{J}$ is a collection of separable subsets of X, then $\tilde{X} := \overline{\mathrm{span}(\cup E)} \in \mathcal{J}$ and $\mathrm{span}(\cup_{X' \in E} A(X'))$ is dense in $A(\tilde{X})$.
Assume also that the same holds with B (resp., $\mathcal{K}$) in place of A (resp., $\mathcal{J}$).

Then, given any $X \in \mathcal{J}$ and $Y \in \mathcal{K}$, any separable subsets $X_0 \subset X$ and $Y_0 \subset Y$, and any linear continuous functions $F : A(X) \to B(Y)$, $G : B(Y) \to A(X)$, there are closed, separable $\tilde{X} \in \mathcal{J}$ and $\tilde{Y} \in \mathcal{K}$ that satisfy $X_0 \subset \tilde{X} \subset X$, $Y_0 \subset \tilde{Y} \subset Y$, $F[A(\tilde{X})] \subset B(\tilde{Y})$ and $G[B(\tilde{Y})] \subset A(\tilde{X})$.

For example, A can stand for L^p for any $p \in [1,\infty)$ if $\mathcal{J}$ is the set of closed (vector) subspaces of a Banach space X (Lemma B.3.15 of [Mik02]). We may even replace $\mathbb{T}$ by any measurable subset of $\mathbb{R}^n$ or by an at most countable set with the counting measure. Alternatively, A can stand for $\mathcal{H}^p$ for any $p \in [1,\infty)$ if $\mathcal{J}$ is the set of closed (vector) subspaces of a Hilbert space X (use Corollary A.7 for "5."). Here we may replace $\mathbb{D}$ by $\mathbb{C}^+$.

Proof of Lemma B.1: (Note: the lemma also holds with the words "vector", "linear" and "span" removed, with the same proof. Alternatively, we need not require the TVSs to be Hausdorff.)

Choose a closed, separable $X_1 \in A(X)$ for X_0 by the property 3.

1° *Finding Y_k:* Given any $k \in \{1, 2, \dots\}$ and a closed, separable $X_k \in \mathcal{J}$ with $X_0 \subset X_k \subset X$, choose a countable dense subset $S_k \subset A(X_k)$. For each $f \in S_k$, choose a closed, separable $Y_f \in \mathcal{K}$ with $Y_0 \subset Y_f \subset Y$ such that $F(f) \in B(Y_f)$ (property 4. for B). Then $Y_k' := Y_0 \cup (\cup_{f\in S_k} Y_f)$ is separable, hence contained in some closed, separable $Y_k \subset Y$ with $Y_k \in \mathcal{K}$.

But $F(f) \in B(Y_k)$ for each $f \in S_k$, hence $F[A(X_k)] \subset B(Y_k)$, by density and continuity.

2° *Finding X_{k+1}:* Similarly, given any $k \in \{1, 2, \dots\}$ and a closed, separable $Y_k \in \mathcal{K}$ with $Y_0 \subset Y_k \subset Y$, we find, as in 1°, a closed, separable $X_{k+1} \in \mathcal{J}$ such that $X_0 \subset X_{k+1} \subset X$ and $G[B(Y_k)] \subset A(X_{k+1})$.

3° Given any sequences of subspaces $X_1, X_2, \dots$ and $Y_1, Y_2, \dots$, chosen as above, set $\tilde{X} := \overline{\mathrm{span}(\cup_k X_k)}$, $\tilde{Y} := \overline{\mathrm{span}(\cup_k Y_k)}$. By "5.", we have $\tilde{X} \in \mathcal{J}$ and $\tilde{Y} \in \mathcal{K}$. Since $G[\cup_k B(Y_k)] \subset \cup_k A(X_k) \subset \tilde{X}$, we have $G[B(\tilde{Y})] \subset \tilde{X}$, by linearity, density and continuity (and "5."). Similarly, $F[A(\tilde{X})] \subset \tilde{Y}$. □

In Theorem 3.2(f), the collection $\mathcal{V}' := \{(A(X), B(Y)) \,|\, (X,Y) \in \mathcal{V}\}$ has properties analogous to those of $\mathcal{V}$:

Lemma B.2 *If the assumptions of Theorem 3.2(f) hold and $\mathcal{V}$ is as in (a), then the spaces $A(X)$ (resp., $B(Y)$) are pairwise orthogonal closed separable subspaces of $A(\mathcal{X})$ (resp., $B(\mathcal{Y})$) and $A(\mathcal{X}) = \sum_{(X,Y)\in\mathcal{V}} A(X)$, $B(\mathcal{Y}) = \sum_{(X,Y)\in\mathcal{V}} B(Y)$. Moreover, $A(\{0\}) = \{0\}$ and $B(\{0\}) = \{0\}$.*

Proof: Now $\mathcal{X} = \overline{\mathrm{span}(\cup_{(X,Y)\in\mathcal{V}} X)}$, hence $\overline{\mathrm{span}(\cup_{(X,Y)\in\mathcal{V}} A(X))}$ is dense in $A(\mathcal{X})$, by "5.", and the spaces $A(X)$ are pairwise orthogonal, by "b." and separable, by "1.". Since $A(\{0\}) \perp A(\{0\})$, by "b.", we have $A(\{0\}) = \{0\}$. □

Proof of Theorem 3.2: We prove the claims roughly in the reverse order with the claim in (b) being (often implicitly) contained in each part of the proof.

(f) Observe first that if $\mathcal{V}$ is of the required form, the claims on f and g (given in (e)) hold, by Lemma B.2 and Theorem 3.1(a1) (because $P_Y^* = P_{B(Y)}^*$, by "b."). The claim on $\mathrm{Ran}(F)$ obviously follows, hence the claim on $\mathrm{rank}(F)$ too. Thus, we only need to find $\mathcal{V}$.

In 3° we shall obtain $\mathcal{V}$ by Hausdorff's Maximality Theorem using the fact that any nonmaximal collection $\tilde{\mathcal{V}}$ of the form specified in 1° can be extended, as will be shown in 2°.

1° *Requirements on* $\tilde{\mathcal{V}}$*:* We require that $\tilde{\mathcal{V}}$ satisfies (a) in place of $\mathcal{V}$ except that $\tilde{\mathcal{X}} := \sum_{(X,Y)\in\mathcal{V}} X$ and $\tilde{\mathcal{Y}} := \sum_{(X,Y)\in\mathcal{V}} X$ need not equal $\mathcal{X}$ and $\mathcal{Y}$, respectively. We also require that

$$F\tilde{P}_{\tilde{\mathcal{X}}} = \tilde{P}_{\tilde{\mathcal{Y}}}F\tilde{P}_{\tilde{\mathcal{X}}} = \tilde{P}_{\tilde{\mathcal{Y}}}F. \tag{71}$$

2° Assume that $\tilde{\mathcal{V}}$ is as in 1° (e.g., $\mathcal{V} = \{(\{0\},\{0\})\}$). Assume also that $\tilde{\mathcal{Y}} \neq \mathcal{Y}$ or $\tilde{\mathcal{X}} \neq \mathcal{X}$ (otherwise $\mathcal{V} := \tilde{\mathcal{V}}$ will do). In 2.1°–2.3° we shall construct closed separable subspaces $X \subset \tilde{\mathcal{X}}^\perp$ and $Y \subset \tilde{\mathcal{Y}}^\perp$, so that $\tilde{\mathcal{V}}' := \tilde{\mathcal{V}} \cup \{(X,Y)\}$ is as in 1° and $X \neq \{0\}$ or $Y \neq \{0\}$.

2.1° *Case* $\tilde{\mathcal{X}} = \mathcal{X}$*:* Pick some $y \in \tilde{\mathcal{Y}}^\perp$ and set $Y := \mathbb{C}y$, $X := \{0\}$ (by (71), $F = F\tilde{P}_{\mathcal{X}} = F\tilde{P}_{\tilde{\mathcal{X}}} = \tilde{P}_{\tilde{\mathcal{Y}}}F$, hence $(I - \tilde{P}_{\tilde{\mathcal{Y}}})F = 0$, hence $\tilde{P}_Y F = 0 = F\tilde{P}_X = \tilde{P}_Y F\tilde{P}_X$).

2.2° *Case* $\tilde{\mathcal{Y}} = \mathcal{Y}$*:* Analogously, pick some $x \in \tilde{\mathcal{X}}^\perp$ and set $X := \mathbb{C}x$, $Y := \{0\}$.

2.3° *Case* $\tilde{\mathcal{X}} \neq \mathcal{X}$ *and* $\tilde{\mathcal{Y}} \neq \mathcal{Y}$*:* Pick some nonempty separable $X_0 \subset \tilde{\mathcal{X}}^\perp$ and $Y_0 \subset \tilde{\mathcal{Y}}^\perp$. Choose X and Y as in Lemma B.1 but with $\tilde{\mathcal{X}}^\perp$, $\tilde{\mathcal{Y}}^\perp$ and F^* in place of $\mathcal{X}$, $\mathcal{Y}$ and G, respectively. Then $FP_X = P_Y^* F P_X$ and $F^* P_Y = P_X^* F^* P_Y$, so

$$F\tilde{P}_X = \tilde{P}_Y^* F\tilde{P}_X = \tilde{P}_Y^* F. \tag{72}$$

Therefore, the requirements in 1° are satisfied for $\tilde{\mathcal{V}}' := \tilde{\mathcal{V}} \cup \{(X,Y)\}$ in place of $\tilde{\mathcal{V}}$:

$$F\tilde{P}_{\tilde{\mathcal{X}}'} = F\tilde{P}_{\tilde{\mathcal{X}}} + F\tilde{P}_X = \tilde{P}_{\tilde{\mathcal{Y}}}F + \tilde{P}_Y F = \tilde{P}_{\tilde{\mathcal{Y}}'}F, \tag{73}$$

hence $\tilde{P}_{\tilde{\mathcal{Y}}'}F\tilde{P}_{\tilde{\mathcal{X}}'} = \tilde{P}_{\tilde{\mathcal{Y}}'}\tilde{P}_{\tilde{\mathcal{Y}}'}F = \tilde{P}_{\tilde{\mathcal{Y}}'}F$.

3° Now we obtain $\mathcal{V}$ by a standard application of Hausdorff's Maximality Theorem. Indeed, let $\mathcal{A}$ the collection of all sets $\tilde{\mathcal{V}}$ that satisfy 1°. Let $\mathcal{A}' \subset \mathcal{A}$ be a maximal subchain and set $\tilde{\mathcal{V}} := \cup\mathcal{A}'$. Then we must have $\tilde{\mathcal{X}} = \mathcal{X}$ and $\tilde{\mathcal{Y}} = \mathcal{Y}$, by maximality (and 2°). Clearly $\mathcal{V} := \tilde{\mathcal{V}}$ satisfies (a).

(e) As noted below Lemma B.1, this is a special case of (f).

(c) 1° *Case* $\mathrm{L}^\infty_{\mathrm{strong}}$*:* By Theorem 4.3, this is a special case of (e) (with $F \in \mathcal{B}(\mathrm{L}^2(\mathcal{X}), \mathrm{L}^2(\mathcal{Y}))$).

2° *Cases* $\mathcal{B}$, $\mathcal{H}^\infty$, $\mathcal{H}^\infty_-$*:* These are subspaces of $\mathrm{L}^\infty_{\mathrm{strong}}$, hence this follows from 1°.

3° *Case* $\mathcal{H}^2_{\mathrm{strong}}$*:* This is a special case of (a), hence it will be established below.

(a) Now the function $G : z \mapsto F(z/2)$ is in $\mathcal{H}^\infty(\mathcal{X},\mathcal{Y})$, so we can choose $\mathcal{V}$ for G, by (c). Fix an arbitrary $(X,Y) \in \mathcal{V}$ for a while. Now

$$\tilde{P}_Y F(z)\tilde{P}_X = \tilde{P}_Y G(2z)\tilde{P}_X = \tilde{P}_Y G(2z) = \tilde{P}_Y F(z) \tag{74}$$

when $|z| < 1/2$, hence $\tilde{P}_Y F(z)\tilde{P}_X = \tilde{P}_Y F(z)$ for each $z \in \mathbb{D}$, by holomorphicity. Therefore, $\tilde{P}_Y F(z)\tilde{P}_X = \tilde{P}_Y F(z)$. Similarly, $\tilde{P}_Y F(z)\tilde{P}_X = F(z)\tilde{P}_X$.

(d) Set $\phi(z) := \sum_k z^k k^{-2}$. Then $\phi \in \mathrm{L}^\infty(\mathbb{C})$, because $\cdot^{-2} \in \ell^1$. Consequently,

$$F * \phi \in \mathcal{B}(\mathcal{X}, \mathrm{L}^\infty(\mathcal{Y})) = \mathrm{L}^\infty_{\text{strong}}(\mathcal{X}, \mathcal{Y}). \tag{75}$$

Pick some $\mathcal{V}$ for $F * \phi \in \mathrm{L}^\infty_{\text{strong}}$ as in (c). Then $G := \sum_{(X,Y)} P_Y^* G_{X,Y} P_X$ is a representative of $F * \phi$, where

$$G_{X,Y} := P_Y(F * \phi)P_X^* \in \mathrm{L}^\infty_{\text{strong}}(X, Y) = \mathrm{L}^\infty(X, Y). \tag{76}$$

Consequently, $\widehat{G}(k) = \sum P_Y^* \widehat{G}_{X,Y}(k) P_X \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ for all k (see Lemma A.5). Moreover, $\widehat{G} = \widehat{F * \phi x} = \widehat{F} x \widehat{\phi}$ on $\mathbb{Z}$, hence

$$\widehat{F} = k^2 \widehat{G}(k) = \sum P_Y^* k^2 \widehat{G}_{X,Y}(k) P_X \quad (k \in \mathbb{Z}). \tag{77}$$

For a while, fix some $(X, Y) \in \mathcal{V}$. Then $\tilde{P}_Y \widehat{F} \tilde{P}_X = \tilde{P}_Y \widehat{F} = \widehat{F} \tilde{P}_X$. By the limit claim in Lemma A.5, we conclude that $Fx = \tilde{P}_Y F x$ a.e. for each $x \in X$, and $\tilde{P}_Y F z = 0$ a.e. for each $z \in X^\perp$, hence $\tilde{P}_Y F = \tilde{P}_Y F \tilde{P}_X = F \tilde{P}_X$ as elements of $\mathrm{L}^1_{\text{strong}}$. The "moreover" claim is obvious.

(g) (Assume that $\mathcal{X} \neq \{0\}$ and $\mathcal{Y} \neq 0$; we omit the other, simpler case.) Apply first (f)2.3° above to these X_0 and Y_0 (or to some of their separable supersets if they are empty) to get a pair, say $(\tilde{X}, \tilde{Y})$. In (f)3°, define $\mathcal{A}$ be the collection of all sets $\tilde{\mathcal{V}}$ that satisfy 1° *and* contain $(\tilde{X}, \tilde{Y})$. The rest of the proofs (of (f) and the others) go as in above. □

Proof of Lemma 3.3: Let $\tilde{\mathcal{V}} := \cup_{j=1}^\infty \mathcal{V}_j \setminus \{\{0\}, \{0\}\}$. Given any $(\tilde{X}, \tilde{Y}) \in \tilde{\mathcal{V}}$, we say that it cuts $(X, Y) \in \tilde{\mathcal{V}}$ if $X \cap X' \neq \{0\}$ or $Y \cap Y' \neq \{0\}$. The elements of $\tilde{\mathcal{V}}$ that lies within a finite chain of cuts from each other obviously form an equivalence class. To be exact, we set $\mathcal{A}_0 := \{(\tilde{X}, \tilde{Y})\}$. Given any $j \in \{1, 2, \ldots\} =: \mathbb{Z}_+$, we set

$$\mathcal{A}_j := \{(X, Y) \in \tilde{\mathcal{V}} \mid X \cap X' \neq \{0\} \text{ or } Y \cap Y' \neq \{0\} \text{ for some } (X', Y') \in \mathcal{A}_{j-1}\}. \tag{78}$$

Obviously, $\mathcal{A}_j \subset \mathcal{A}_{j+1}$ for each j. The equivalence class of $(\tilde{X}, \tilde{Y})$ is then $\mathcal{A} := \cup_j \mathcal{A}_j$. By Theorem 3.1(a3), it is at most countable.

For each $\mathcal{A}$, the spaces $X_\mathcal{A} := \sum_{(X,Y) \in \mathcal{A} \cap \mathcal{V}_1} X$, $Y_\mathcal{A} := \sum_{(X,Y) \in \mathcal{A} \cap \mathcal{V}_1} Y$ are closed, separable subspaces, so we can define $\mathcal{V}$ to be the collection of such pairs $(X_\mathcal{A}, Y_\mathcal{A})$ (obviously, $X_\mathcal{A} = Y_\mathcal{A}$ for each class $\mathcal{A}$ if $X = Y$ for each j and each $(X, Y) \in \mathcal{V}_j$).

If $(X, Y) \in \mathcal{V}_j$ for some j, and $\mathcal{A}$ is the class of (X, Y), then $X \perp X'$ and $Y \perp Y'$ for each $(X', Y') \in \cup_j \mathcal{V}_j \setminus \mathcal{A}$, hence for each $(X', Y') \in \mathcal{V}_1 \setminus \mathcal{A}$, hence $X' \subset X_\mathcal{A}$ and $Y' \subset Y_\mathcal{A}$. We conclude that

$$X_\mathcal{A} = \cup\{X \in \mathcal{V}_j^\mathcal{X} \mid X \in \mathcal{A}\} = \cup\{X \in \mathcal{V}_j^\mathcal{X} \mid X \cap X_\mathcal{A} \neq \{0\}\} \tag{79}$$

for each j and $\mathcal{A}$, where $\mathcal{V}_j^\mathcal{X} := \{X \mid (X, Y) \in \mathcal{V}_j \text{ for some } Y \subset \mathcal{Y}\}$. Since each $Y_\mathcal{A}$ has similar properties, we observe that $\mathcal{V}$ has the required properties. E.g., if $F \sum_X \tilde{P}_X = \sum_X F \tilde{P}_X$ and $\sum_Y P_Y F = \sum_Y \tilde{P}_Y F$ for countable sums of separable orthogonal subspaces and $\tilde{P}_Y F = F\tilde{P}_X$ for each $(X,Y) \in \mathcal{V}_j$ for some fixed j, then

$$FP_{X_\mathcal{A}} = F \sum_{(X,Y)\in\mathcal{V}_j^\mathcal{X}\cap\mathcal{A}} \tilde{P}_X = \sum_{\mathcal{V}_j^\mathcal{X}\cap\mathcal{A}} F\tilde{P}_X = \sum_{\mathcal{V}_j^\mathcal{X}\cap\mathcal{A}} \tilde{P}_Y F = \tilde{P}_{Y_\mathcal{A}} F \tag{80}$$

for each class $\mathcal{A}$, i.e., for each $(X_\mathcal{A}, Y_\mathcal{A}) \in \mathcal{V}$. □

C $\mathrm{L}^\infty_{\mathrm{strong}}$ and inner functions

In this section we prove Theorems C.1 and 4.10 and illustrate some pathologies of $\mathrm{L}^\infty_{\mathrm{strong}}$ over nonseparable Hilbert spaces in three examples.

If, e.g., $F \in \mathrm{L}^q_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$ (resp., $F : \mathbb{D} \to \mathcal{B}(\mathcal{X},\mathcal{Y})$), then the $\mathcal{B}(\mathrm{L}^p)$ norm (resp., the $\mathcal{B}(\mathcal{H}^p)$ norm) of the multiplication operator $M_F : f \mapsto Ff$ equals $\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} \le \infty$:

Theorem C.1 *Assume that X and Y are Banach spaces, $X \ne \{0\}$, and $1 \le p \le \infty$. If F is a linear map from X to functions $\mathbb{T} \to Y$, then*

$$\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \sup_{0\ne g\in\mathcal{P}(\mathbb{C})X} \frac{\|Fg\|_p}{\|g\|_p} \le \infty. \tag{81}$$

If F is a function $\mathbb{D} \to \mathcal{B}(X,Y)$, then

$$\|F\|_{\mathcal{H}^\infty} = \sup_{0\ne g\in\mathcal{P}(\mathbb{C})X} \frac{\|Fg\|_{\mathcal{H}^p}}{\|g\|_{\mathcal{H}^p}} \le \infty. \tag{82}$$

We can replace $\mathcal{P}(\mathbb{C})X$ by $\mathrm{L}^p(X)$ in (81) and by $\mathcal{H}^p(X)$ in (82).

Note that if $F \in \mathcal{H}^q_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$, $f \in \mathcal{P}(\mathbb{C})$ and $x \in \mathcal{X}$, then $\|Ffx\|_{\mathcal{H}^p} = \|(Fx)_0 f\|_{\mathrm{L}^p}$, where $(Fx)_0 \in \mathrm{L}^q$ denotes the boundary function of $Fx \in \mathcal{H}^q(\mathcal{Y})$ (indeed, because $Ffx \in \mathcal{H}^q$ and Ffx converges to $(Fx)_0 f$ a.e. on $\mathbb{T}$, Ffx is the Poisson integral of $(Fx)_0 f$, by Proposition 2.3).

Proof: 1° *Proof of (81):* Assume that $F : \mathcal{X} \to (\mathbb{T} \to \mathcal{Y})$ is linear. If $Fx \notin \mathrm{L}^p(\mathcal{Y})$ for some x, then all terms of (81) equal ∞, so we may assume that $F \in \mathrm{L}^p_{\mathrm{strong}}(\mathcal{X},\mathcal{Y})$. Since "$\ge$" is obvious in (81) and case $\|F\| = 0$ is trivial, we may assume that $\|F\| > 0$, and, given $M \in (0, \|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}})$, we only need to find $g \in \mathcal{P}(\mathbb{C})\mathcal{X}$ with $\|Fg\|_p > M\|g\|_p$.

1.1° Choose $x \in \mathcal{X}$ so that $\|x\| = 1$ and $M < \|Fx\|_\infty$. Set $G := Fx \in \mathrm{L}^p(\mathcal{Y})$. Then $\|G\|_\infty > M$, hence $m(A) > 0$, where $A := \{z \in \mathbb{T} \mid \|G(z)\|_\mathcal{Y} > M\}$.

If $p = \infty$, then we can take $g \equiv x$, so assume that $p < \infty$. Let z_0 be a Lebesgue point of χ_A. Set

$$f_r := |z - rz_0|^{-1} \in \mathcal{H}^p(\mathbb{C}), \qquad g_r := f_r/\|f_r\|_p \qquad (r > 1). \tag{83}$$

Choose $\epsilon > 0$ such that $M_\epsilon^p := M^p(1-\epsilon-\epsilon/2) > M^p$. Choose then $\tilde{R} > 0$ such that $m(B_R \setminus A)/m(B_R) < \epsilon$ for every $R \in (0, \tilde{R}]$, where $B_R := \{z \in \mathbb{T} \,\big|\, |z - z_0| < R\}$ (p. 141 of [Rud87]). For each $r > 1$, set $B := B_{\tilde{R}}$,

$$c_r := \inf\{g_r(z) \,\big|\, z \in B\}. \tag{84}$$

As $r \to 1+$, we have $\|f_r\|_p \to \infty$ and $f_r(z)$ is bounded by $|z - z_0|^{-1}$, hence $g_r(z) \to 0$, for each $z \neq z_0$, hence $c_r \to 0$. Moreover,

$$g_r(z) > c_r \Leftrightarrow z \in B \quad (z \in \mathbb{T}), \tag{85}$$

hence there exists $r > 1$ such that

$$\|\chi_{B^c} g_r\|_p^p < \epsilon/2. \tag{86}$$

But for any $t \geq c_r$, there exists $R_t \in [0, \tilde{R}]$ such that

$$\{g_r > t\} := \{z \in \mathbb{T} \,\big|\, g_r(z) > t\} = B_{R_t}, \tag{87}$$

hence $m\{\chi_{A^c} g_r > t\} = m(A^c \cap B_{R_t}) \leq \epsilon m(B_{R_t}) = \epsilon m\{g_r > t\}$. By this, (85) and Theorem 8.16 of [Rud87], we have

$$\begin{aligned} \int_{B\setminus A} g_r(z)^p \, dz &= \int_{c_r}^\infty m\{\chi_{A^c} g_r > t\} p t^{p-1} \, dt \\ &\leq \epsilon \int_{c_r}^\infty m\{g_r > t\} p t^{p-1} \, dt \leq \epsilon \|g_r\|_p^p = \epsilon. \end{aligned} \tag{88}$$

This and (86) imply that $\|\chi_{A\cap B} g_r\|_p^p > 1 - \epsilon - \epsilon/2$. With the definition of A, this leads to

$$\|G g_r\|_p^p > \|\chi_{A\cap B} G g_r\|_p^p \geq M^p(1 - \epsilon - \epsilon/2) = M_\epsilon^p > M^p. \tag{89}$$

By Lemma A.4, $|K_n * g_r - g_r| \to 0$ uniformly on $\mathbb{T}$, as $n \to \infty$, hence

$$\int_{\mathbb{T}} \|G[K_n * g_r - g_r]\|_{\mathcal{Y}}^p \, dm = \int_{\mathbb{T}} \|G\|_{\mathcal{Y}}^p |K_n * g_r - g_r|^p \, dm \to 0. \tag{90}$$

This and (89) imply that $a_n^p := \int_{\mathbb{T}} \|G(K_n * g_r)\|_{\mathcal{Y}}^p \, dm > M^p$ for n big enough. But $b_n := \|K_n * g_r\|_p \to \|g_r\|_p = 1$, as $n \to \infty$, hence $a_n > M b_n$ for n big enough. For $g := K_n * g_r x$ we have

$$\|Fg\|_p = \|G(K_n * g_r)\|_p = a_n > M b_n = M \|K_n * g_r\|_p = M \|g\|_p. \tag{91}$$

The norms of Fg and g (on $\mathbb{T}$) are unaffected if we multiply g by z^n, and then $g \in \mathcal{P}(\mathbb{C})x \subset \mathcal{P}(\mathbb{C})\mathcal{X}$.

2° *Proof of (82):* Assume that $F \in \mathcal{H}^p_{\mathrm{strong}}$ (otherwise both sides of (82) are infinite). Since "$\geq$" is obvious, we only need to find, for an arbitrary M such that $M < \|F\|_{H^\infty}$, a polynomial $g \in \mathcal{P}(\mathbb{C})X$ such that $\|Fg\|_{\mathcal{H}^p} > M\|g\|_p$.

Since $\|F(z)\|_{\mathcal{B}(X,Y)} > M$ for some $z \in \mathbb{D}$, there exist $\Lambda \in Y^*$ and $x \in X$ such that $\|\Lambda\| = 1 = \|x\|$ and $|G(z)| > M$, where $G := \Lambda F x \in \mathcal{H}^\infty(\mathbb{C},\mathbb{C}) = \mathcal{H}^\infty(\mathbb{D};\mathbb{C})$. In particular, $M < |G(z)| \le \|G\|_{\mathcal{H}^\infty} = \|G\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \|G\|_\infty$. But $\|Gg\|_{\mathcal{H}^p} = \|Gg\|_p$ for each $g \in \mathcal{P}(\mathbb{C})$, hence, by 1°, there exists $\tilde{g} \in \mathcal{P}(\mathbb{C})$ such that $\|G\tilde{g}\|_{\mathcal{H}^p} > M\|\tilde{g}\|_{\mathcal{H}^p}$. Set $g := \tilde{g}x$ to have

$$\|Fg\|_{\mathcal{H}^p(Y)} \ge \|\Lambda F g\|_{\mathcal{H}^p(\mathbb{C})} = \|G\tilde{g}\|_{\mathcal{H}^p(\mathbb{C})} > M\|g\|_{\mathcal{H}^p(X)}. \tag{92}$$

3° The last claim follows: Obviously, "$\ge$" still holds, but the supremum over a bigger set cannot be less either. □

Proof of Theorem 4.10: Fix a representative F of F as in Corollary 3.5.

1° *(a):* For each $(X,Y) \in \mathcal{V}$. we have $(F^*F)_{X,X} = F^*_{X,Y}F_{X,Y}$, by Theorem 3.1(a2) (set $Z_X := X$), hence $(I - F^*F)_{X,X} = I_{X,X} - F^*_{X,Y}F_{X,Y}$. By Theorem 3.1, $I - F^*F = 0$ iff $(I - F^*F)_{X,X} = 0$ (in $\mathrm{L}^\infty_{\mathrm{strong}}(X,X)$) for each $(X,Y) \in \mathcal{V}$, hence (a) holds.

2° *(b):* If $\mathcal{X}$ is separable and $\tilde{F} \in F$, then, by Lemma 3.6, $F^*F = I$ iff $\tilde{F}(z)^*\tilde{F}(z) = I$ a.e. The last claim will be established in Example C.3.

3° *We have (i)⇔(ii)⇔(iii):* Obviously, (iii) is a reformulation of (ii), and we have (i)⇒(ii')⇒(ii). Assume then (ii). Find $\mathcal{V}$ for F as in Theorem 3.2 Then, by (b), for each $(X,Y) \in \mathcal{V}$ we find a null set $N_X \subset \mathbb{T}$ such that for each $z \in \mathbb{T} \setminus N_X$ we have $\langle x, F^*(z)F(z)x\rangle = \|x\|^2$ for each $x \in X$. Therefore, $P_X F^*(z)F(z)x = x$ for each $x \in X$. By (a), we obtain (i).

4.1° *We have (iii)⇒(vi)⇒(v):* From Proposition 2.3 we get "(vi)⇒(v)". If $f : \mathbb{T} \to \mathcal{X}$ is measurable, then it is almost separably-valued, so outside a null set its range is contained in a closed, separable subspace $X_0 \subset \mathcal{X}$. If (iii) holds, then there exists a null set $N \subset \mathbb{T}$ such that $\|F(z)x\| = \|x\|$ for all $z \in \mathbb{T} \setminus N$ and $x \in X_0$, consequently, then $\|Ff\|_p = \|f\|_p \le \infty$.

4.2° *We have (v)⇒(iii):* Let $x \in \mathcal{X}$ be arbitrary. If $\|Fx\|_{\mathcal{Y}} > \|x\|_{\mathcal{X}}$ on a set of positive measure, then $\|Ff\|_p > \|f\|_{\mathcal{H}^p}$ for some $f \in \mathcal{P}(\mathbb{C})\mathcal{X}$, by Theorem C.1. If $\|F(z)x\| \le \|x\|$ for a.e. $z \in \mathbb{T}$ but not $\|Fx\| = \|x\|$ a.e., then we obviously have $\|Ff\|_p < \|f\|_p$, where $f(z) := x$ for all $z \in \mathbb{T}$, hence then (v) does not hold. Because x was arbitrary, "not (iii)" implies "not (v)", i.e., (v) implies that (iii) holds.

5° *We have (ii)⇐(vii)⇔(viii)⇐(i):* Implications (viii)⇒(vii)⇒(ii) are obvious, so assume that (i) holds. Pick $\mathcal{V}$ for F as in Theorem 3.2(c). By (a) and (b), each $F_{X,Y}$ has a representative $\tilde{F}_{X,Y}$ such that $\tilde{F}_{X,Y}(z)^*\tilde{F}_{X,Y}(z) = I$ for each $z \in \mathbb{T}$ (redefine the one from (b) on a null set). By Theorem 3.1, $\tilde{F} := \sum_{\mathcal{V}} \tilde{F}_{X,Y}$ satisfies $[\tilde{F}] \in \mathrm{L}^\infty_{\mathrm{strong}}$, $[\tilde{F}] = F$, and $\tilde{F}(z)^*\tilde{F}(z) = I$ for each $z \in \mathbb{T}$ (by the case $\mathcal{B}$ applied to each $\tilde{F}(z) \in \mathcal{B}(\mathcal{X};\mathcal{Y})$), hence (viii) holds.

6° *We have (iv)⇔(iii):* We obviously, have (vi)⇒(iv)⇒(v) for $p = 2$. But (iii)⇔(vi)⇔(v), by 4°.

7° *(c):* If Fx is not measurable for some $x \in \mathcal{X}$, then $Fx = Fg$ is not measurable, where $g \equiv x$. Thus, if (v) holds, then $\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = 1 < \infty$, by Theorem C.1, hence then $F^*F = I$, by 4.2°&3°.

8° *(d):* If G is inner, then $\|Gf\|_{\mathcal{H}^p} = \|f\|_{\mathcal{H}^p}$ for each $f \in \mathcal{H}^p(\mathcal{X})$, by "(i)$\Leftrightarrow$(v)" (4°). Conversely, if $\|Gf\|_{\mathcal{H}^p} = \|f\|_{\mathcal{H}^p}$ for each $f \in \mathcal{P}(\mathbb{C})\mathcal{X}$, then $\|G\|_{\mathcal{H}^\infty} = 1$, by Theorem C.1, and (v) holds, hence then G is inner, by 4°.

9° *(e):* This follows from (viii) and Lemma A.1(b). □

We may have $[F] = 0 \in \mathrm{L}^\infty_{\mathrm{strong}}$ even if $F(z) \neq 0 \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ for each $z \in \mathbb{T}$:

Example C.2 **(a)** Assume that $\mathcal{X} = \ell^2(\mathbb{T}; \mathbb{C})$ and $\mathcal{Y} \neq \{0\}$. Pick $y_0 \in Y$ such that $\|y_0\| = 1$. For each $z \in \mathbb{T}$, define $\Lambda_z \in \mathcal{X}^*$ by $\Lambda_z x := x(z)$ $(x \in \mathcal{X})$ and $F(z) \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ by

$$F(z)x := zy_0\Lambda_z x = zx(z)y_0. \tag{93}$$

(Note that $\|F(z)\| = |z| = 1$ for each $z \in \mathbb{T}$.) Given $x \in \mathcal{X}$, we have $F(z)x = 0$ a.e., hence $\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = 0$ even though $F(z) \neq 0$ for each $z \in \mathbb{T}$.

(b) Note also that $F(z)^* y_0 = \bar{z} e_z$, where $\langle x, e_z \rangle = x(z)$. Therefore, $F(z)^* y_0$ is not measurable (not being almost separably-valued), so F^* is not strongly measurable (hence not $\mathrm{L}^\infty_{\mathrm{strong}}$).

(c) If we replace y_0 by $(\operatorname{Re} z)^{-1} y_0$ in (93), then $\|F(z)\|_{\mathcal{B}(\mathcal{X},Y)} = |\operatorname{Re} z|^{-1}$, hence then $\operatorname{ess\,sup} \|F\|_{\mathcal{B}} = \infty$ even though still $[F] = [0] \in \mathrm{L}^\infty_{\mathrm{strong}}$ and $F^* y_0$ is nonmeasurable. ◁

Even worse, we may have $F \in \mathcal{H}^\infty$ inner with boundary function $F_0 \in \mathrm{L}^\infty_{\mathrm{strong}}$ such that $Fx \to F_0 x$ nontangentially at every point of $\mathbb{T}$, for every $x \in \mathcal{X}$ and yet $F_0(z)^* F_0(z) \neq I$ for each $z \in \mathbb{T}$. Indeed the boundary function of the function $h \in \mathcal{H}^\infty$ given by $h(z) := \mathrm{e}^{(z+1)/(z-1)}$ satisfies $h(1) = 0$ and $|h(z)| = 1$ for $z \in \mathbb{T} \setminus \{1\}$ (by Lemma 6.1). By rotating h by all possible angles and combining these uncountably many rotated copies to a function $F \in \mathcal{H}^\infty(\mathcal{X})$, this function has the properties explained above:

Example C.3 Define $h : \mathbb{D} \to \mathbb{C}$ by $h(z) := \mathrm{e}^{(z+1)/(z-1)}$. Then $h \in \mathcal{H}^\infty$ is inner, $h(z) = \exp(-2i \operatorname{Im} z/|z-1|^2) \in \mathbb{T}$ for $z \in \mathbb{T} \setminus \{1\}$, and $h(1) = 0$ (and all these limits are nontangential).

Set $\mathcal{X} := \ell^2(\mathbb{T}; \mathbb{C})$, and define $F : \overline{\mathbb{D}} \to \mathcal{B}(\mathcal{X})$ by $(Fe_s)(z) := h(\bar{s}z)e_s$ for each $s \in \overline{\mathbb{D}}$, where $e_s := \chi_{\{s\}}$ (the functions e_s form the natural orthonormal basis of $\mathcal{X}$). Then F is inner, by Theorem 4.10(a) (set $\mathcal{V} := \{(X_s, X_s) \bigm| s \in \mathbb{T}\}$, where $X_s := \mathbb{C}e_s$, so that $F_{X,X} x = hx$ $(x \in X)$ for each $(X, X) \in \mathcal{V}$).

Moreover, $F_0 := F|_{\mathbb{T}}$ is the unique function $\mathbb{T} \to \mathcal{B}(\mathcal{X})$ for which $F_0 x$ is the (nontangential) limit of $F|_{\mathbb{D}} x$ for each $x \in \mathcal{X}$. Nevertheless, $F(z)^* F(z) = I - P_z \neq I$ for each $z \in \mathbb{T}$, where P_z is the orthogonal projection $\mathcal{X} \to \mathbb{C}e_z$. ◁

Even in the above example, we could redefine F_0 (within the same class in $\mathrm{L}^\infty_{\mathrm{strong}}$) so that $F_0(z)^* F_0(z) = I$ for each $z \in \mathbb{T}$ (e.g., by setting above $h(1) := 1$), by Theorem 4.10(viii). However, then $F_0(z)e_z = e_z$ would no longer be equal to the nontangential limit 0 of Fe_z at z, for any $z \in \mathbb{T}$. Thus, the fact that $F_0(z)^*F_0(z) \neq I$ everywhere is inherent in the inner function F, not a consequence of an artificial choice of F_0 within $[F_0]$.

Finally, we now show that the condition $\sup_{\mathbb{T}} \|f_{X,Y}\|_{\mathcal{B}(X,Y)} = \|F_{X,Y}\|_{\mathrm{L}^\infty_{\text{strong}}}$ in Corollary 3.5 is not extraneous: in the following example only this condition is violated and the resulting function f is not $\mathcal{B}(\mathcal{X},\mathcal{Y})$-valued.

Example C.4 Let $\mathcal{X} = \mathcal{Y} = \ell^2(\mathbb{N}\times\mathbb{T})$. For each $(n,z) \in \mathbb{N}\times\mathbb{T}$ we set $\mathcal{V} = \{(X_{n,z}, X_{n,z}) \mid (n,z) \in \mathbb{N}\times\mathbb{T}\}$, $X_{n,z} := \mathbb{C}e_{n,z}$, $f_{n,z} := nT_{n,z}\chi_{\{z\}} \in \mathrm{L}^\infty_{\text{strong}}(X_{n,z}, X_{n,z})$, where the canonical basis element $e_{n,z} \in \mathcal{X}$ is defined by

$$e_{n,z}(k,w) := \begin{cases} 1, & \text{when } k = n,\ w = z; \\ 0, & \text{otherwise,} \end{cases} \tag{94}$$

$T_{n,z} \in \mathcal{B}(X_{n,z}, X_{n,z})$ is the multiplication by $e_{n,z}$ and χ is the characteristic function.

Then the assumptions of Corollary 3.5 (except the sup-condition) are satisfied for $F = 0 \in \mathrm{L}^\infty_{\text{strong}}(\mathcal{X},\mathcal{X})$, but $\|f(z)\|_{\mathcal{B}} \geq \sup_n \|f_{n,z}(z)\| = \sup_n n = \infty$ for every $z \in \mathbb{T}$, so f is not $\mathcal{B}(\mathcal{X},\mathcal{Y})$-valued. ◁

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