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Nonlinear potential theory of elliptic equations with nonstandard growth

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Teemu Lukkari: *Nonlinear potential theory of elliptic equations with nonstandard growth*; Helsinki University of Technology, Institute of Mathematics, Research Reports A541 (2008).

Abstract: *We consider the nonlinear potential theory of elliptic partial differential equations with nonstandard structural conditions. In such a theory, Harnack inequalities and the class of superharmonic functions related to the equation under consideration have a crucial role. We develop a technique for proving Harnack type inequalities to handle possibly unbounded solutions. After this, we show that the basic properties of the related superharmonic functions are similar to the case of standard structural conditions, and give applications of Harnack inequalities and superharmonicity. These include removability, growth of fundamental solutions, and superharmonic functions as solutions of equations involving measures.*

AMS subject classifications: 35J70, 35J60, 31C45, 35B45, 35B50, 35D05, 35D10, 46E35

Keywords: Nonstandard growth, variable exponent, $p(x)$-Laplacian, logarithmic Hölder continuity, Caccioppoli estimate, Moser iteration, Harnack's inequality, regularity, comparison principle, superharmonic function, removability, growth of solutions, existence of generalized solutions, measure data

Teemu Lukkari: *Epälineaarinen potentiaaliteoria elliptisille yhtälöille, joissa esiintyy epästandardeja rakenne-ehtoja*; Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja A541 (2008).

Tiivistelmä: *Käsittelemme epästandardeja rakenne-ehtoja sisältävien elliptisten osittaisdifferentiaaliyhtälöiden potentiaaliteoriaa. Tälläisessa teoriassa Harnack-tyyppisillä epäyhtälöillä ja käsiteltävään yhtälöön liittyvillä superharmonisilla funktioilla on ratkaiseva rooli. Kehitämme tekniikan Harnack-estimaattien todistamiseen mahdollisesti rajoittamattomille ratkaisuille. Tämän jälkeen näytämme, että superharmonisten funktioiden perusominaisuudet ovat samankaltaiset kuin standardien rakenne-ehtojen tapauksessa, ja käsittelemme Harnack-estimaattien ja superharmonisuuden sovelluksia. Näihin kuuluvat poistuvuus, fundamentaaliratkaisujen kasvu ja superharmoniset funktiot mittadataa sisältävien yhtälöiden ratkaisuina.*

Avainsanat: Epästandardi rakenne-ehto, varioiva eksponentti, $p(x)$-Laplacen yhtälö, logaritminen Hölder-jatkuvuus, Caccioppoli-estimaatti, Moserin iteraatio, Harnackin epäyhtälö, säännöllisyys, vertailuperiaate, superharmoninen funktio, poistuvuus, ratkaisujen kasvuvauhti, yleistettyjen ratkaisujen olemassaolo, mittadata

Preface

The work presented in this dissertation was carried out during the years 2005-2007, at the Institute of Mathematics, Helsinki University of Technology. Financial support has been provided by the Academy of Finland, and the Finnish Academy of Science and Letters.

The interest and support of my advisor Juha Kinnunen during the process is very much appreciated. I am also grateful to Petri Juutinen and Fumi-Yuki Maeda for acting as the pre-examiners of the dissertation.

My collaborators, Petteri Harjulehto Peter Hästö, Mika Koskenoja and Niko Marola, have earned my sincere gratitude by their efforts. In particular, docent Harjulehto provided me with some of his notes early on. Without them, getting started would have been considerably harder.

Working at the Institute of Mathematics has been a pleasant experience. The credit for this goes to my co-workers, and the exceptional spirit of the nonlinear PDE group.

I would also like to thank my family and all of friends for providing that crucial something else. Finally, a quite special thanks goes to all the nice folks at Club Anvil. Lemma 3.4 in [I]– essentially the crux of this thesis – came to me after a night spent under the mirrorball.

Espoo, February 2008 Teemu Lukkari

List of included articles

The current dissertation consists of this overview and the following publications.

[I] P. Harjulehto, J. Kinnunen, and T. Lukkari. Unbounded supersolutions of nonlinear equations with nonstandard growth. *Bound. Value Probl.*, 2007:Article ID 48348, 20 pages, 2007. Available at `http://www.hindawi.com/GetArticle.aspx?doi=10.1155/2007/48348`.

[II] P. Harjulehto, P. Hästö, M. Koskenoja, T. Lukkari, and N. Marola. An obstacle problem and superharmonic functions with nonstandard growth. *Nonlinear Anal.* 67(12):3424–3440, 2007.

[III] T. Lukkari. Singular solutions of elliptic equations with nonstandard growth. To appear in *Math. Nachr.* Preprint available at `http://mathstat.helsinki.fi/analysis/varsobgroup/`.

[IV] T. Lukkari. Elliptic equations with nonstandard growth involving measures. To appear in *Hiroshima Math. J.* Preprint available at `http://mathstat.helsinki.fi/analysis/varsobgroup/`.

In the text, these publications are referred to by the roman numerals above.

Author's contribution

The work presented in this dissertation was carried out during the years 2005-2007, at the Institute of Mathematics, Helsinki University of Technology.

In [I], the author had a key role in the analysis, and did most of the writing.

In [II], the author's contribution is mainly the relationship between $p(x)$-superharmonic functions and supersolutions and the summability properties of $p(x)$-superharmonic functions.

Publications [III] and [IV] report independent research by the author.

1 Introduction

The topic of this thesis is the $p(x)$-laplacian

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = 0, \tag{1.1}$$

and the related nonlinear potential theory. Equations similar to (1.1) arise as Euler-Lagrange equations of variational integrals like

$$\int |\nabla u|^{p(x)}\,\mathrm{d}x. \tag{1.2}$$

Equation (1.1) exhibits growth of '$p(x)$-type', which is a particular class of so-called *nonstandard growth conditions.*

Classical potential theory concerns the properties of solutions of the Laplace equation

$$-\operatorname{div}(\nabla u) = 0. \tag{1.3}$$

The term potential theory stems from physics, namely from the fact that the scalar potential u of a static electric field satisfies (1.3). See [13] for a recent account of the potential theory related to (1.3). The canonical example of a solution of (1.3) in three-dimensional euclidean space is the function

$$u(x) = |x|^{-1}, \tag{1.4}$$

the potential of a point charge at the origin. Observe that this function does not belong to the natural Sobolev space $W^{1,2}$ associated to (1.3); instead, it belongs to a strictly larger class called superharmonic functions.

In the mid-eighties, *p-superharmonic functions* were introduced [59]; this class consists of lower semicontinuous functions that obey the comparison principle with respect to continuous solutions of the equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0. \tag{1.5}$$

See also [47, 48]. An example of such a function in the n-dimensional Euclidean space is given by

$$u(x) = |x|^{-(n-p)/(p-1)}. \tag{1.6}$$

When $p = 2$ and $n = 3$, this reduces to the potential of a point charge given by (1.4). As an alternative, equivalent formulation of (1.5), one may consider the minimisers of the p-Dirichlet energy integral

$$\int |\nabla u|^p\,\mathrm{d}x.$$

The class of p-superharmonic functions coincides with the viscosity supersolutions of (1.5) by [52].

There are structural differences between the equations (1.3), (1.5) and (1.1). The Laplace equation is linear, i.e. if u and v are solutions, then $\lambda u+\mu v$, where λ and μ are constants, is also a solution. The p-laplacian is no longer linear, but it is nevertheless scalable; $u+v$ is usually not a solution, but $\lambda u+\mu$ is still a solution. The $p(x)$-laplacian (1.1) in turn exhibits a stronger form of nonlinearity, since, in general, $\lambda u+\mu$ is no longer a solution if $\lambda \neq \pm 1$. These structural differences are reflected in potential theory in a fundamental way. In the case of (1.3), one usually employs representation formulas, such as the mean value property, for solutions. When considering (1.5), such formulas need to be replaced by estimates, for instance Caccioppoli type estimates, and various weak Harnack inequalities. Such estimates for (1.5) date back to the sixties, [74, 76]. Finally, as one passes from (1.5) to (1.1), versions of these estimates continue to hold. However, the estimates become intrinsic, i.e. they depend on the solution itself. This intrinsicness turns out to be a major source of difficulties.

Equations with energies like (1.1) appear in the modelling of electrorheological fluids. One of the features of such fluids is the fact that the mechanical properties of the fluid are influenced by an external electric field. One approach to model this phenomenon is to introduce an extra stress tensor with a variable exponent, [69, 71]. Another application for minimising the integral (1.2) is image processing [19, 58]; see also [45]. In this field, it is desirable to choose the exponent $p(x)$ so that it takes the value 1 near the edges in an image, and the value 2 in "smoothly" varying regions. The first of these properties helps in preserving the sharpness of the edges, and the second removes noise from the smoothly varying parts of an image.

The contribution of this dissertation is twofold. First, we modify Moser's iteration technique so that we can prove Harnack estimates needed in potential theory without the boundedness assumptions required in earlier literature. Removing the boundedness restriction is essential, since the examples (1.4) and (1.6) show that we need estimates that apply to unbounded functions. Second, we present applications of such estimates. These include the basic properties of $p(x)$-superharmonic functions, removable singularities and growth of singular solutions, and equations with measure-valued right hand side. In particular, we prove the existence and estimate the growth of the $p(x)$-counterpart of (1.6).

2 Partial differential equations involving variable growth exponents

The minimisation of (1.2) was first considered by Zhikov, starting in the mid-eighties [78]. To get a feel of the problems involved, we shall give some examples introduced by him. To this end, let us denote

$$\begin{aligned} F(x,\xi) &= |\xi|^{p(x)},\ V(\Omega) = \{u \in W_0^{1,1}(\Omega) : \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x < \infty\},\ \text{and}\\ \widetilde{V}(\Omega) &= \{u \in V(\Omega) :\ \text{there is } u_\varepsilon\ \in C_0^\infty(\Omega) \text{ such that } u_\varepsilon \to u \text{ in } W^{1,1}(\Omega)\\ &\qquad \text{and } \lim_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^{p(x)}\,\mathrm{d}x = \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x\}. \end{aligned}$$

In Zhikov's terminology [80], F is called *regular* if

$$V(\Omega) = \widetilde{V}(\Omega).$$

In particular, when F is *not* regular, it is possible that

$$E = \min_{u\in W_0^{1,1}(\Omega)} \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x < \inf_{u\in C_0^\infty(\Omega)} \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x = \widetilde{E};$$

this is the *Lavrentiev phenomenon*.

Example 2.1. Let $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ and

$$p(x) = \begin{cases} \alpha & \text{for } x_1x_2 > 0,\\ \beta & \text{for } x_1x_2 \le 0, \end{cases}$$

where $1 < \alpha < 2 < \beta$. Then F is not regular, [78, 80].

Example 2.2. Let $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ and

$$p(x) = \begin{cases} \alpha & \text{for } x_1 > 0,\\ \beta & \text{for } x_1 \le 0. \end{cases}$$

Then F is regular, [78, 80].

For the exponent of Example 2.1, it can be shown that the minimiser of (1.2) is not continuous, and does not possess the higher integrability property, [80]. The discontinuity is not essential, as shown by Hästö [50]; there is a uniformly continuous exponent such that the minimiser is not continuous.

The above examples show that some kind of assumption on the exponent $p(\cdot)$ is necessary to develop a regularity theory for (1.1). Zhikov has introduced the *logarithmic Hölder continuity condition* (2.3) for this purpose. It is usually stated in the form of a logarithmic modulus of continuity, i.e.

$$|p(x) - p(y)| \le C\frac{1}{-\log|x-y|}, \tag{2.3}$$

where the inequality is required to hold for points x and y such that $|x-y| \le 1/2$. Indeed, the condition (2.3) is sufficient for regularity of F, as shown in [79].

2.1 The variable exponent Lebesgue and Sobolev spaces

One way to see that (2.3) implies the regularity of F is to consider the density of smooth functions in a suitable "energy space" related to the functional (1.2). The first step in defining such a space is to define a variable exponent Lebesgue space. Roughly speaking, this space $L^{p(\cdot)}$ consists of all measurable functions u such that $|u|^{p(x)} \in L^1$. The Luxemburg norm, defined as

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \int_\Omega |u/\lambda|^{p(x)}\,\mathrm{d}x \leq 1\},$$

turns $L^{p(\cdot)}$ into a Banach space. This and other basic properties of $L^{p(\cdot)}$ were established by in the early nineties by Kováčik and Rákosník [54]; see also [37]. For exponents $p(\cdot)$ such that

$$1 < \inf p(x) \leq \sup p(x) < \infty, \tag{2.4}$$

these properties are very similar to the classical L^p space. In particular, the dual of $L^{p(\cdot)}$ is obtained by conjugating the exponent pointwise. Hence $L^{p(\cdot)}$ is reflexive, and bounded sets are weakly compact. This fact is useful in applications to partial differential equations. The main difference is that $L^{p(\cdot)}$ is almost never translation invariant, i.e. if $f \in L^{p(\cdot)}$, it does not follow that $x \mapsto f(x+h) \in L^{p(\cdot)}$ when $h \neq 0$. As a consequence, approximation using the usual convolution procedure becomes involved.

Now it is possible to define the variable exponent Sobolev space in the obvious way. The space $W^{1,p(\cdot)}$ consists of functions $u \in L^{p(\cdot)}$ whose distributional gradient ∇u exists and belongs to $L^{p(\cdot)}$. This space is a Banach space with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$$

If smooth functions are dense in $W^{1,p(\cdot)}$, then the corresponding integrand F is regular, in particular the Lavrentiev phenomenon does not occur. To see this, note that if $f_i \to f$ in $L^{p(\cdot)}$, then $|f_i|^{p(x)} \to |f|^{p(x)}$ in L^1_{loc}. Density allows us to pick a sequence of smooth functions (u_i) such that $u_i \to u$ in $W^{1,p(\cdot)}$, and an application of the previous observation to $|\nabla u_i|^{p(x)}$ yields the regularity of F.

The key results related to density of smooth functions are due to Samko [72] and Diening [25]. Samko showed that if $p(\cdot)$ is log-Hölder continuous, then averaging by convolution is bounded on $L^{p(\cdot)}$. This in turn implies the density of smooth functions. Diening has discovered the underlying general principle by showing that if the Hardy-Littlewood maximal function, defined as

$$Mf(x) = \sup_{B \ni x} \fint_B |f(y)|\,\mathrm{d}y,$$

is bounded on $L^{p(\cdot)}$, smooth functions are dense in $W^{1,p(\cdot)}$. See also [30, 51]. Further, log-Hölder continuity of the exponent is sufficient for M to be bounded on $L^{p(\cdot)}$ [25] and examples show [68] that (2.3) is sharp in a certain sense. More precisely, if we insist that the assumptions on $p(\cdot)$ are given terms of an estimate for the modulus of continuity, then the condition (2.3) is sharp. Note that boundedness on $L^{p(\cdot)}$ implies that a norm estimate,

$$\|Mf\|_{p(\cdot)} \le C\|f\|_{p(\cdot)},$$

holds, not that

$$\int_\Omega (Mf)^{p(x)}\,\mathrm{d}x \le C\int_\Omega |f|^{p(x)}\,\mathrm{d}x. \tag{2.5}$$

Indeed, it can be shown that (2.5) holds if and only if $p(\cdot)$ is constant, [56].

The maximal operator M on $L^{p(\cdot)}$ has been widely studied and applied; see, e.g., [24, 26, 27, 57, 28, 29, 23, 22]. The theory of the spaces $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ can also be used for showing the existence of minimisers of (1.2), or equivalently solutions of (1.1). For instance, one can apply the direct method of the calculus variations under mild conditions on $p(\cdot)$ and the boundary data, as shown in [44]. See also [35, 46].

2.2 Regularity of solutions

Weak solutions of (1.1) can be defined in the usual way; a function $u \in W^{1,p(\cdot)}(\Omega)$ is a solution if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\,\mathrm{d}x = 0$$

for all test functions $\varphi \in C_0^\infty(\Omega)$. However, the Caccioppoli estimates needed to prove regularity require usage of test functions that depend on the solution itself. Fortunately, the dual of $L^{p(\cdot)}$ is obtained by conjugating the exponent pointwise; Thus, one can allow test functions $\varphi \in W_0^{1,p(\cdot)}(\Omega)$ by an approximation argument, provided that smooth functions are dense in $W^{1,p(\cdot)}$.

Logarithmic Hölder continuity (2.3) is also sufficient for the Hölder continuity of solutions of (1.1). This follows from De Giorgi type estimates, as shown by Fan and Zhao [36], or from Harnack's inequality by Alkhutov [7]. Acerbi and Mingione [2] have developed a freezing technique specifically for functionals similar to (1.2). Their technique allows one to prove the Hölder continuity of minimisers given (2.3), and also the Hölder continuity of the gradient of a minimiser if $p(\cdot)$ itself is Hölder continuous. See also [20, 21, 31, 32, 8].

The role of the assumption (2.3) in regularity results can be briefly illustrated as follows. In [36, 7], Hölder continuity follows from estimates for the supremum of the solution in balls. The constants appearing in such estimates must be independent of the radius R of the ball, at least for small values of R. In order to achieve this, the quantity

$$R^{-(p^+-p^-)},$$

where $p^+ = \max_{x\in B_R} p(x)$ and $p^- = \min_{x\in B_R} p(x)$, needs to be controlled. Hence the oscillation $p^+ - p^-$ of the exponent needs vanish fast enough, so that

$$R^{-(p^+-p^-)} \le C. \tag{2.6}$$

It turns out that (2.6) is equivalent with (2.3). The condition (2.6) also appears in the estimates of this thesis. Diening [25, Lemma 3.2] has given a proof of the equivalence.

The regularity results discussed above are sharp in the same sense as the boundedness of the maximal function, i.e. in terms of the modulus of continuity of $p(\cdot)$. Indeed, in Hästö's example of a discontinuous minimiser [50], the modulus of continuity of the exponent is only slightly worse than that allowed by (2.3). Further, it can be shown that the gradient of a minimiser is Hölder continuous only if the exponent is also Hölder continuous. This follows from an explicit formula for the minimiser of (1.2) on the real line, [42, Corollary 5.2.].

For further regularity results, see [1, 38, 10, 5, 3, 4, 34, 18, 9]. Evolution equations related to (1.1) are considered, e.g., in [11, 77, 12, 49, 6].

3 Potential theory with nonstandard growth

3.1 Harnack estimates for unbounded solutions and supersolutions

Moser iteration and Harnack's inequality for nonnegative solutions of (1.1) have been previously considered by Alkhutov [7]; see also [55, 9]. The local boundedness of solutions is first established by a preliminary iteration, and then the actual Moser iteration is carried out. This results in

$$\sup_{x\in B_R} u(x) \le C(u)(\inf_{x\in B_R} u(x) + R), \tag{3.1}$$

where the constant depends on the supremum of u, i.e.

$$C(u) \approx 1 + (\sup_{x\in B_{4R}} u(x))^{p^+-p^-}, \tag{3.2}$$

where $p^- = \inf_{x\in B_{4R}} p(x)$ and $p^+ = \sup_{x\in B_{4R}} p(x)$.

Difficulties in the iteration procedure arise due to the fact that there are no natural "$p(x)$-Sobolev inequalities" available. Hence passing between a constant exponent in a Sobolev inequality to a variable exponent in an estimate obtained from the equation and back needs to be taken care of. The first step is handled by Young's inequality and considering $v = u + R$ instead of u. This results in the extra term R on the right hand side of (3.1). The second step is handled by using the boundedness of u in the estimates, and results in (3.2).

We improve this result in [I]. The improvement is based on the observation that if $p(\cdot)$ is log-Hölder continuous, then integral averages of the form

$$\fint_{B_r} f^{p^+_{B_r}-p^-_{B_r}}\,\mathrm{d}x$$

can be controlled by L^t norms of f for small values of t. This trick is inserted into the latter step, i.e. passing back from a variable exponent to a constant one. These modifications lead to the inequality (3.1), but now with

$$C(u) \approx 1 + (\|u\|_{L^t(B_{4R})})^{p^+ - p^-}. \tag{3.3}$$

The values of t are restricted only by the requirement that $t > (p^+ - p^-)/q'$, where q is any number such that $1 < q < n/(n-1)$ and q' is the Hölder conjugate of q. Thus, since $p(\cdot)$ is assumed to be continuous, t can be chosen to be arbitrarily small by considering sufficiently small balls. This is particularly useful in nonlinear potential theory, where the estimate

$$\left(\fint_{B_{2R}} u^{q_0} \, \mathrm{d}x \right)^{1/q_0} \leq C(u)(\inf_{x \in B_R} u(x) + R), \tag{3.4}$$

the *weak Harnack inequality for nonnegative supersolutions* is a crucial tool.

To illustrate the origin of the dependence (3.3), consider the variational integral

$$\int_\Omega F(x, \nabla u) \, \mathrm{d}x,$$

where the density F satisfies the *standard* structural condition

$$\alpha |\xi|^p \leq F(x, \xi) \leq \beta |\xi|^p.$$

It is well known that estimates for minimisers depend only on the ellipticity ratio β/α, not on the minimiser itself. The structural condition for the functional (1.2) is

$$\alpha |\xi|^{p(x)} \leq F(x, \xi) \leq \beta |\xi|^{p(x)},$$

and a "worst case" of this is

$$\alpha |\xi|^{p^-} \leq F(x, \xi) \leq \beta (|\xi|^{p^+} + 1), \tag{3.5}$$

where $p^- = \min p(x)$ and $p^+ = \max p(x)$. Thus, by formally computing the ellipticity ratio corresponding to (3.5), one would expect estimates to depend on $(\beta/\alpha) \times |\nabla u|^{p^+ - p^-}$. Further, note that Moser's method uses Caccioppoli estimates, i.e. estimates for the gradient of a solution in terms of the solution itself. This allows us to replace a dependence on $|\nabla u|$ by a dependence on u. Hence the qualitative form of (3.3) should not be surprising.

As an application of the improved estimate (3.4), we study the infinity set of $p(x)$-superharmonic functions. More precisely, we prove that if $u \in L^t_{loc}(\Omega)$ is a lower semicontinuous function such that $\min(u, \lambda)$ is a supersolution of (1.1) for each $\lambda \in \mathbb{R}$, then

$$C_{p(\cdot)}(\{x \in \Omega : u(x) = \infty\}) = 0.$$

Here $C_{p(\cdot)}$ is the Sobolev $p(x)$-capacity, i.e.

$$C_{p(\cdot)}(E) = \inf \int_{\mathbb{R}^n} \left(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \right) \mathrm{d}x,$$

where infimum is taken over the set of admissible functions

$$S_{p(\cdot)}(E) = \{u \in W^{1,p(\cdot)}(\mathbb{R}^n) : u \geq 1 \text{ in an open set containing } E\}.$$

We refer to [43, 41] for details on the various $p(x)$-capacities.

We also provide an example which shows that the constant in (3.1) cannot be independent of u. The example uses the explicit formula for minimizers of (1.2) on the real line, [42].

Antontsev and Zhikov [11] obtain a parabolic version of the supremum estimate by using Moser iteration. Their argument is similar to the preliminary iteration in [7].

The condition (3.5) can be considered a special form of the (p, q)*-growth condition*

$$\alpha|\xi|^p \leq F(x, \xi) \leq \beta(|\xi|^q + 1), \tag{3.6}$$

where $p < q$. There is an extensive literature on variational functionals and partial differential equations with (p, q)-growth, starting from the late eighties; see for example [64, 63, 1, 40, 33, 39, 67], and the references in the survey [66].

3.2 The definition and properties of $p(x)$-superharmonic functions

In the constant exponent case, p-superharmonic functions are defined as the lower semicontinuous functions which obey the comparison principle with respect to solutions. More spesifically, a function $u : \Omega \to (-\infty, \infty]$ is p-*superharmonic in* Ω, if

1. u is lower semicontinuous,
2. u is not identically ∞ in any component of Ω, and
3. the comparison principle holds: Let $D \Subset \Omega$ be an open set. If h is a solution in D, continuous in $\overline{D}$ and $u \geq h$ on ∂D, then $u \geq h$ in D.

The weak Harnack inequality for unbounded supersolutions in [I] makes developing the theory of similarly defined $p(x)$*-superharmonic functions* feasible. We undertake this task in [II], along the lines of Lindqvist [59], Heinonen and Kilpeläinen [47] and the monograph [48].

The basic tools needed to develop the theory of superharmonic functions are existence and regularity results for the obstacle problem, and convergence theorems for supersolutions and solutions. These tools are used to approximate $p(x)$-superharmonic functions with supersolutions of (1.1).

In the obstacle problem, an obstacle function $\psi : \Omega \to [-\infty, \infty)$, and boundary data $w \in W^{1,p(\cdot)}(\Omega)$ are given, and then one tries to find the minimal supersolution above the obstacle, and with the desired boundary values. More specifically, we let

$$\mathcal{K}_{\psi,w}(\Omega) = \big\{u \in W^{1,p(\cdot)}(\Omega) \colon u - w \in W_0^{1,p(\cdot)}(\Omega), u \geq \psi \text{ a.e. in } \Omega\big\},$$

and say that a function $u \in \mathcal{K}_{\psi,w}(\Omega)$ is a *solution of the obstacle problem in* $\mathcal{K}_{\psi,w}(\Omega)$ if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla(v-u)\, dx \geq 0$$

for every test function $v \in \mathcal{K}_{\psi,w}(\Omega)$. The existence of a solution follows by a convexity argument.

Regularity for the obstacle problem consists of the fact that the solution of the obstacle problem is continuous whenever the obstacle is continuous, and of the fact that whenever the minimal supersolution does not touch the obstacle, it is a solution. The first fact requires a Harnack estimate, and in [II] we establish this estimate in a fashion similar to [I].

We also prove the classical convergence theorems for increasing and decreasing sequences of supersolutions, and for uniformly convergent sequences of solutions. All these results are quite straightforward adaptations of arguments found, for example, in [48]. This is not case with Harnack's principle, which states that the limit of an increasing sequence of solutions is either a solution or identically infinite. This cannot be true in the variable exponent setting as such; the constant in Harnack's inequality (3.1) depends on the solution itself. Our version of Harnack's principle requires the additional assumption that the limit function is in L^t_{loc} for some $t > 0$.

Estimates for superharmonic functions require a stronger relation to the equation (1.1) than the one provided by the comparison principle, even for constant exponents. Such a link is provided by the fact that u is p-superharmonic if and only if the truncations $\min(u,k)$ are supersolutions. The obstacle problem is the tool one needs to establish this characterisation, [59], and in [II] we show that this property is carried over to the $p(x)$-superharmonic functions; the relationship between $p(x)$-superharmonic functions and supersolutions of (1.1) is same as in the case of (1.5). Further, this relationship is exploited to show that for smooth, radial exponents, the function

$$v(x) = \int_{|x|}^1 (p(r)r^{n-1})^{-1/(p(r)-1)}\, \mathrm{d}r \tag{3.7}$$

is $p(x)$-superharmonic.

The final topic we cover in [II] is the summability (i.e. integrability) of $p(x)$-superharmonic functions. The main result is that given an open set $U \subset \Omega$ and a compact set $K \subset U$, a $p(x)$-superharmonic function u and its gradient Du belong to $L^{q(p(x)-1)}(K)$ for some $q > 1$, if $u \in L^t(U \setminus K)$ for some $t > 0$. Here, Du is the generalised gradient of u, defined pointwise as

$$Du = \lim_{k\to\infty} \nabla \min(u,k). \tag{3.8}$$

Note that Du is not necessarily the weak gradient u, since it is possible that $u \notin W^{1,1}_{loc}$.

In our summability result, we use a method based on estimates for the truncations $\min(u,k)$. This method was applied to estimate p-superharmonic functions by Kilpeläinen and Malý [53]; see also [15, 16]. First, u is mollified in $U \setminus K$ by using the Poisson modification. The truncations of the mollified version of u belong to $W_0^{1,p(\cdot)}(U)$, and this fact is then used to prove the required estimates. Carrying out the Poisson modification requires Harnack's principle and boundary regularity results. The assumption $u \in L^t(U \setminus K)$ is required by (3.3) to make Harnack's principle work, and the necessary boundary regularity results are due to Alkhutov and Krasheninnikova [9].

3.3 Removable and nonremovable singularities of solutions

The Moser iteration and Harnack's inequality can be used to prove removability theorems, and to estimate the growth of solutions near a nonremovable isolated singularity. This was established by Serrin [74, 75] for a general class of elliptic equations similar to the p-laplacian (1.5). In [III] we extend the theory to cover the equation (1.1).

A removability theorem asserts that a function u which is a solution in $\Omega \setminus E$ can be extended so that it becomes a solution in Ω given additional assumptions on the compact set E and on u. In Serrin's removability theorem, the s-capacity of E is assumed to be zero for some $s \geq p$. In order to conclude the desired extension property from this, the solution u needs to be integrable to a power that depends on s and p. In [III] we generalise this so that the assumption on E is given in terms of a variable exponent capacity and u is assumed to be integrable to the variable power which corresponds pointwise to the constant exponent case. For this purpose, it is essential that the iteration from [I] applies to unbounded solutions, since boundedness is not assumed in the removability theorem.

Serrin also considered solutions with a nonremovable isolated singularity at the origin in [74]. Such a solution must have exactly the same growth rate as the fundamental solution (1.6) of the p-laplacian, i.e.

$$C_1|x|^{-(n-p)/(p-1)} \leq u(x) \leq C_2|x|^{-(n-p)/(p-1)}, \tag{3.9}$$

where C_1 and C_2 are constants. The proof is based on estimating the growth in terms of p-capacity of suitable balls by using Harnack's inequality. The bounds in (3.9) then follow by computing the capacity of a ball in terms of its radius.

In our setting, the crucial point is Harnack's inequality, due to (3.3). Thus we need L^p estimates for singular solutions. It turns out that if we assume that

$$\lim_{x \to 0} u(x) = \infty,$$

it follows that u is $p(x)$-superharmonic. Then we use the summability result from [II] to prove that our solution u is locally integrable to a small power. After this, an estimate for u in terms of $p(\cdot)$-capacity follows.

The final tool we need is a way to estimate the $p(\cdot)$-capacity of balls. In the constant exponent case, such estimates use the fundamental solution of the p-laplacian; see for example [62, Theorem 2.8]. However, there is no general formula for the fundamental solution of (1.1). Nevertheless, using an idea due to Alkhutov and Krasheninnikova [9], we are able to compute a useful lower bound for the $p(\cdot)$-capacity. This bound implies an estimate similar to (3.9).

3.4 Equations with measure-valued right hand side

We have seen above that some properties of $p(x)$-superharmonic functions require the additional assumption that $u \in L^t_{loc}$ for some $t > 0$. It would be desirable to have a way of showing that such functions exist. We accomplish this in [IV] by showing that if μ is a positive, finite Borel measure on Ω, then there is an integrable $p(x)$-superharmonic function u such that

$$-\operatorname{div}\mathcal{A}(x, Du) = \mu \tag{3.10}$$

in the sense of distributions. Here $\mathcal{A}$ is an operator with properties similar to the $p(x)$-laplacian in (1.1), and Du is the generalised gradient of u defined by (3.8). For related results, see [60, 73, 70].

The proof of the result mimics the one for the constant exponent case by Kilpeläinen and Malý [53] and Mikkonen [65]. There are two stages in the proof. First, one establishes a compactness result; out of any sequence of $p(x)$-superharmonic functions one can extract a limit which is $p(x)$-hyperharmonic, i.e. satisfies (1) and (3), but not necessarily (2), in the definition of $p(x)$-superharmonic functions . Second, the original measure μ is approximated in the sense of weak convergence by more regular measures μ_j, so that one can find supersolutions u_j satisfying

$$-\operatorname{div}\mathcal{A}(x, \nabla u_j) = \mu_j.$$

Then the compactness theorem is applied to find the solution of (3.10). However, some care is required to show first that the the solution found is actually integrable. This approach is related to the works of Boccardo and Gallouët [15, 16]; see also [17, 61, 14].

As an application, we show that by taking $\mu = \delta$ in (3.10), one obtains a solution with a nonremovable isolated singularity, the counterpart of (3.7). Due to (3.3), showing the existence of such solutions is not possible by the methods of Serrin [74].

References

[1] E. Acerbi and N. Fusco. A transmission problem in the calculus of variations. *Calc. Var. Partial Differential Equations*, 2:1–16, 1994.

[2] E. Acerbi and G. Mingione. Regularity results for a class of functionals with nonstandard growth. *Arch. Rational Mech. Anal.*, 156:121–140, 2001.

[3] E. Acerbi and G. Mingione. Regularity results for a class of quasiconvex functionals with nonstandard growth. *Ann. Scuola Norm. Sup. Cl. Sci. IV*, 30:311–339, 2001.

[4] E. Acerbi and G. Mingione. Regularity results for stationary electro-rheological fluids. *Arch. Ration. Mech. Anal.*, 164(3):213–259, 2002.

[5] E. Acerbi and G. Mingione. Gradient estimates for the $p(x)$-Laplacean system. *J. reine ang. Math.*, 584:117–148, 2005.

[6] E. Acerbi, G. Mingione, and G. A. Seregin. Regularity results for parabolic systems related to a class of non-Newtonian fluids. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21(1):25–60, 2004.

[7] Yu. A. Alkhutov. The Harnack inequality and the Hölder property of solutions of nonlinear elliptic equations with a nonstandard growth condition. *Differential Equations*, 33(12):1653–1663, December 1997.

[8] Yu. A. Alkhutov. On the Hölder continuity of $p(x)$-harmonic functions. *Mat. Sb.*, 196(2):3–28, 2005.

[9] Yu. A. Alkhutov and O. V. Krasheninnikova. Continuity at boundary points of solutions of quasilinear elliptic equations with nonstandard growth. *Izv. Ross. Akad. Nauk. Ser. Mat.*, 68(6):3–60, 2004. Translation in *Izv. Math.* **68** (2004), no. 6, 1063-1117.

[10] S. Antontsev and S. Shmarev. Elliptic equations and systems with nonstandard growth conditions: Existence, uniqueness and localization properties of solutions. *Nonlinear Analysis*, 65:728–761, 2006.

[11] S. Antontsev and V. Zhikov. Higher integrability for parabolic equations of $p(x,t)$-Laplacian type. *Adv. Differential Equations*, 10:1053–1080, 2005.

[12] S. N. Antontsev and S. I. Shmarev. A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions. *Nonlin. Anal.*, 60:515–545, 2005.

[13] D. H. Armitage and S. J. Gardiner. *Classical Potential Theory*. Springer, 2001.

[14] L. Boccardo, A. Dall'Aglio, T. Gallouët, and L. Orsina. Nonlinear parabolic equations with measure data. *J. Funct. Anal.*, 147(1):237–258, 1997.

[15] L. Boccardo and T. Gallouët. Non-linear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, 87:149–169, 1989.

[16] L. Boccardo and T. Galloüet. Non-linear elliptic equations with right hand side measures. *Comm. Partial Diff. Eq.*, 17:641–655, 1992.

[17] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vazquez. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 22:241–274, 1995.

[18] V. Bögelein and A. Zatorska-Goldstein. Higher integrability of very weak solutions of systems of $p(x)$-Laplacean type. *J. Math. Anal. Appl.*, 336(1):480–497, 2007.

[19] Y. Chen, S. Levine, and M. Rao. Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.*, 66(4):1383–1406 (electronic), 2006.

[20] V. Chiadò Piat and A. Coscia. Hölder continuity of minimizers of functionals with variable growth exponent. *Manuscripta Math.*, 93(3):283–299, 1997.

[21] A. Coscia and G. Mingione. Hölder continuity of the gradient of $p(x)$-harmonic mappings. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(4):363–368, 1999.

[22] D. Cruz-Uribe and A. Fiorenza. Approximate identities in variable L^p spaces. *Math. Nachr.*, 280(3):256–270, 2007.

[23] D. Cruz-Uribe, A. Fiorenza, J. M. Martell, and C. Perez. The boundedness of classical operators on variable L^p spaces. *Ann. Acad. Sci. Fenn. Math.*, 31:239–264, 2006.

[24] D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer. The maximal function on variable L^p spaces. *Ann. Acad. Sci. Fenn. Math.*, 28:223–238, 2003.

[25] L. Diening. Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. *Math. Inequal. Appl.*, 7(2):245–253, 2004.

[26] L. Diening. Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces. *Bull. Sci. Math.*, 129:657–700, 2005.

[27] L. Diening and M. Ružička. Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics. *J. Reine Angew. Math.*, 563:197–220, 2003.

[28] L. Diening and M. Ružička. Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$. I. *J. Math. Anal. Appl.*, 298(2):559–571, 2004.

[29] L. Diening and M. Ružička. Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$. II. *J. Math. Anal. Appl.*, 298(2):572–588, 2004.

[30] D. E. Edmunds and J. Rákosník. Density of smooth functions in $W^{k,p(x)}(\Omega)$. *Proc. Roy. Soc. London Ser. A*, 437(1899):229–236, 1992.

[31] M. Eleuteri. Hölder continuity results for a class of functionals with nonstandard growth. *Boll. Unione Mat. Ital. (8)*, 7-B:129–157, 2004.

[32] M. Eleuteri. Regularity results for a class of obstacle problems. *Appl. Math.*, 52(2):137 – 170, 2007.

[33] L. Esposito, F.Leonetti, and G. Mingione. Sharp regularity for functionals with (p, q) growth. *J. Diff. Equations*, 204:5–55, 2004.

[34] X. Fan. Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form. *J. Diff. Equations*, 235(2):397–417, 2007.

[35] X. Fan and Q. Zhang. Existence of solutions for $p(x)$-Laplacian Dirichlet problem. *Nonlinear Anal.*, 52(8):1843–1852, 2003.

[36] X. Fan and D. Zhao. A class of De Giorgi type and Hölder continuity. *Nonlinear Anal.*, 36(3):295–318, 1999.

[37] X. Fan and D. Zhao. On the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. *J. Math. Anal. Appl.*, 263:424–446, 2001.

[38] X. Fan and D. Zhao. Regularity of quasiminimizers of integral functionals with discontinuous $p(x)$-growth conditions. *Nonlinear Anal.*, 65:1521–1531, 2006.

[39] I. Fonseca, J. Malý, and G. Mingione. Scalar minimizers with fractal singular sets. *Arch. Ration. Mech. Anal.*, 172(2):295–307, 2004.

[40] N. Fusco and C. Sbordone. Higher integrability of the gradient of minimizers of functionals with nonstandard growth conditions. *Comm. Pure Appl. Math.*, 43(5):673–683, 1990.

[41] P. Harjulehto, P. Hästö, and M. Koskenoja. Properties of capacities in variable exponent Sobolev spaces. *J. Anal. Appl.* 5(2):71–92, 2007.

[42] P. Harjulehto, P. Hästö, and M. Koskenoja. The Dirichlet energy integral on intervals in variable exponent Sobolev spaces. *Z. Anal. Anwend.*, 22(4):911–923, 2003.

[43] P. Harjulehto, P. Hästö, M. Koskenoja, and S. Varonen. Sobolev capacity on the space $W^{1,p(\cdot)}(\mathbb{R}^n)$. *J. Funct. Spaces Appl.*, 1:17–33, 2003.

[44] P. Harjulehto, P. Hästö, M. Koskenoja, and S. Varonen. The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values. *Potential Anal.*, 25(3):205–222, 2006.

[45] P. Harjulehto, P. Hästö, and V. Latvala. Minimizers of the variable exponent, non-uniformly convex Dirichlet energy. *J. Math. Pures Appl. (9)*, 89(2):174–197, 2008.

[46] P. Hästö. On the existence of minimizers of the variable exponent Dirichlet energy integral. *Commun. Pure Appl. Anal.*, 5(3):413–420, 2006.

[47] J. Heinonen and T. Kilpeläinen. A-superharmonic functions and supersolutions of degenerate elliptic equations. *Ark. Mat.*, 26(1):87–105, 1988.

[48] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations.* Oxford University Press, 1993.

[49] E. Henriques and J. M. Urbano. Intrinsic scaling for pdes with an exponential nonlinearity. *Indiana Univ. Math. J.*, 55:1701–1722, 2006.

[50] P. Hästö. Counter examples of regularity in variable exponent Sobolev spaces. *Contemp. Math.*, 367:133–143, 2005.

[51] P. Hästö. On the density of continuous functions in variable exponent Sobolev space. *Rev. Mat. Iberoamericana*, 23(1):215–237, 2007.

[52] P. Juutinen, P. Lindqvist, and J. J. Manfredi. On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation. *SIAM J. Math. Anal.*, 33(3):699–717 (electronic), 2001.

[53] T. Kilpeläinen and J. Malý. Degenerate elliptic equations with measure data and nonlinear potentials. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19:591–613, 1992.

[54] O. Kováčik and J. Rákosník. On spaces $L^{p(x)}$ and $W^{1,p(x)}$. *Czechoslovak Math. J.*, 41(116):592–618, 1991.

[55] O. V. Krasheninnikova. On the continuity at a point of solutions of elliptic equations with a nonstandard growth condition. *Tr. Mat. Inst. Steklova*, 236(Differ. Uravn. i Din. Sist.):204–211, 2002.

[56] A. K. Lerner. On modular inequalities in variable L^p spaces. *Arch. Math.*, 85:538–543, 2005.

[57] A. K. Lerner. Some remarks on the Hardy-Littlewood maximal function on variable L^p spaces. *Math. Z.*, 251(3):509–521, 2005.

[58] S. Levine. An adaptive variational model for image decomposition. In *Energy Minimization Methods in Computer Vision and Pattern Recognition, 5th International Workshop*, volume 3757 of *Lecture Notes in Computer Science*, pages 382–397, St. Augustine, FL, USA, 2005. Springer Verlag.

[59] P. Lindqvist. On the definition and properties of p-superharmonic functions. *J. Reine Angew. Math.*, 365:67–79, 1986.

[60] F-Y. Maeda. $p(x)$-harmonic functions with isolated singularities. *RIMS Kôkyûroku*, 1553:116-131, Kyoto University, 2007

[61] F-Y. Maeda. Renormalized solutions of Dirichlet problems for quasilinear elliptic equations with general measure data. *Hiroshima Math. J.*, to appear.

[62] J. Malý and W. P. Ziemer. *Fine Regularity of Solutions of Elliptic Partial Differential Equations.* American Mathematical Society, 1997.

[63] P. Marcellini. Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions. *Arch. Rational Mech. Anal.*, 105(3):267–284, 1989.

[64] P. Marcellini. Regularity and existence of solutions of elliptic equations with p, q-growth conditions. *J. Differential Equations*, 90(1):1–30, 1991.

[65] P. Mikkonen. On the Wolff potential and quasilinear elliptic equations involving measures. *Ann. Acad. Sci. Fenn. Math. Diss.*, 104:1–71, 1996.

[66] G. Mingione. Regularity of minima: an invitation to the dark side of the calculus of variations. *Appl. Math.*, 51(4):355–425, 2006.

[67] G. Mingione and D. Mucci. Integral functionals and the gap problem: Sharp bounds for relaxation and energy concentration. *SIAM J. Math. Anal.*, 36(5):1540–1579, 2005.

[68] L. Pick and M. Ružička. An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded. *Expo. Math.*, 19:369–371, 2001.

[69] K. R. Rajagopal and M. Ružička. Mathematical modeling of electrorheological materials. *Cont. Mech. and Thermodyn*, 13:59 – 78, 2001.

[70] J. F. Rodrigues, M. Sanchón, and J. M. Urbano. The obstacle problem for nonlinear elliptic equations with variable growth exponents and L^1-data. Preprint.

[71] M. Ružička. *Electrorheological Fluids: Modeling and Mathematical Theory.* Springer, 2000.

[72] S. Samko. Denseness of $C_0^\infty(\mathbf{R}^N)$ in the generalized Sobolev spaces $W^{M,P(X)}(\mathbf{R}^N)$. In *Direct and inverse problems of mathematical physics (Newark, DE, 1997)*, volume 5 of *Int. Soc. Anal. Appl. Comput.*, pages 333–342. Kluwer Acad. Publ., Dordrecht, 2000.

[73] M. Sanchón and J. M. Urbano. Entropy solutions for the $p(x)$-laplace equation. *Trans. Amer. Math. Soc.* To appear.

[74] J. Serrin. Local behavior of solutions of quasi-linear equations. *Acta Math.*, 111:247–302, 1964.

[75] J. Serrin. Isolated singularities of solutions of quasi-linear equations. *Acta Math.*, 113:219–240, 1965.

[76] N. S. Trudinger. On Harnack type inequalities and their application to quasilinear elliptic equations. *Comm. Pure Appl. Math.*, 20:721–747, 1967.

[77] M. Xu and Y. Z. Chen. Hölder continuity of weak solutions for parabolic equations with nonstandard growth conditions. *Acta Math. Sin. (Engl. Ser.)*, 22(3):793–806, 2006.

[78] V. V. Zhikov. Averaging of functions of the calculus of variations and elasticity theory. *Math. USSR-Izv.*, 29(1):33–66, 1987. Translation of Izv. Akad. Nauk. SSSR Ser. Math. 50 (1986), no. 4, 675–710.

[79] V. V. Zhikov. On Lavrentiev's phenomenon. *Russian J. Math. Phys.*, 3(2):249–269, 1995.

[80] V. V. Zhikov. On some variational problems. *Russian J. Math. Phys.*, 5(1):105–116, 1997.

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