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Abstract: *Given any bounded operator T in a Banach space X we discuss simple approximations for the resolvent $(\lambda - T)^{-1}$, rational in λ and polynomial in T. We link the convergence speed of the approximation to the Green's function for the outside of the spectrum of T and give an application to computing Riesz projections.*

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1 Introduction

Let T be a bounded operator in a complex Banach space X and denote by $\sigma(T)$ its spectrum. The *resolvent* of T is the analytic operator valued function

$$\lambda \mapsto (\lambda - T)^{-1} \tag{1.1}$$

defined in the *resolvent set* $\rho(T) = \mathbb{C} \setminus \sigma(T)$. For λ with $|\lambda|$ large enough the series

$$(\lambda - T)^{-1} = \sum_{j=0}^{\infty} T^j \lambda^{-1-j} \tag{1.2}$$

converges and can be used e.g. in holomorphic functional calculus, but there is no "general formula" for the resolvent which could be used everywhere outside the spectrum.

An important feature of (1.2) is that the operator T enters only in *positive* powers. In fact, then the solution of

$$(\lambda - T)x = f \tag{1.3}$$

exists in the invariant subspace $\mathcal{K}(T, f)$ obtained as the closure of all vectors of the form $p(T)f$ where p is a polynomial. In numerical analysis the finite dimensional subspaces

$$\mathcal{K}_n(T, f) = \mathrm{span}_{0 \le j < n}\{T^j f\} \tag{1.4}$$

are called *Krylov* subspaces and the convergence of *Krylov solvers* is hence naturally linked with polynomial approximation of the resolvent. It follows from the maximum principle that we can hope to approximate the resolvent by polynomials in T only in the *unbounded* component of the complement of the spectrum, that is for $\lambda \notin \hat{\sigma}(T)$. Recall that if $F \subset \mathbb{C}$ is compact, then

$$\hat{F} = \{\lambda \in \mathbb{C} \ : \ |p(\lambda)| \le \max_{z \in F} |p(z)| \ \text{ for all polynomials } p\} \tag{1.5}$$

denotes the *polynomially convex hull* of F and is obtained by "filling in" the possible holes in F.

In this paper we give a simple procedure to express the resolvent for $\lambda \notin \hat{\sigma}(T)$. This is based on the notion of *polynomial numerical hull* $V(T)$ of the operator T which we introduced in [11], see Section 2. By Theorem 2.10.3 in [11] (or Theorem 9.4.6 in [3]) we have always

$$V(T) = \hat{\sigma}(T). \tag{1.6}$$

Since $\lambda_0 \notin V(T)$ is equivalent with the existence of a polynomial p such that

$$|p(\lambda_0)| > \|p(T)\| \tag{1.7}$$

we use such a polynomial to give us a representation of the resolvent near λ_0 in terms of a locally converging power series.

With this representation we can improve our treatment of convergence speeds for (GMRES-like) Krylov methods, see in particular Theorem 6.2 and Remark 6.5 below. In fact, in [11] we divided the convergence mechanisms into *sublinear, linear* and *superlinear* convergence. The sublinear phenomenon was treated in the spirit of a discrete analytic semigroup, while the linear phase was modelled by problems with large spectrum and dealt with potential theory outside the spectrum. It was clear that the convergence would ultimately be superlinear for all "right hand sides" f in (1.3) if and only if $\sigma(T)$ has zero capacity (that is, T is *quasialgebraic*, [8]). However, quantitative estimates were worked out only under the extra assumption that the resolvent was *meromorphic* for $\lambda \neq 0$. Operators with meromorphic resolvents were further discussed in [13]. A good source on quasialgebraicity containing classical results on polynomial approximation of analytic functions in compact sets is [14].

2 Polynomial numerical hull of a bounded operator

Denote by $\mathbb{P}_k$ the set of complex polynomials of degree not exceeding k and by $\mathbb{P}$ the set of all polynomials. Given $T \in B(X)$ we put, see [11]

$$V^k(T) = \{\lambda \in \mathbb{C} \ : \ |p(\lambda)| \leq \|p(T)\| \ \text{ for all } p \in \mathbb{P}_k\} \tag{2.1}$$

and

$$V(T) = \cap_{k \geq 1} V^k(T). \tag{2.2}$$

We call $V(T)$ the *polynomial numerical hull* of T and $V^k(T)$ the *polynomial numerical hull* of T of *degree* k. These form a nonincreasing sequence of nonempty compact sets. Additionally, if

$$W(T) = \{x^*(Tx) \ : \|x\| = 1, x^* \in j(x)\}$$

is the *numerical range* of T, where $j(x) = \{x^* \in X^* \ : x^*(x) = \|x\|^2 = \|x^*\|^2\}$, then $V^1(T)$ is the convex closure of $W(T)$ while, see (1.6), $V(T)$ equals $\hat{\sigma}(T)$, the polynomially convex hull of the spectrum, [11]. For later works, see [12], [6], [5], [7], [4], [2], [1], [3], [15].

Suppose now that $\lambda \notin \hat{\sigma}(T)$ (or $\lambda \notin V^d(T)$) so that there exists a polynomial p (of degree at most d) such that

$$|p(\lambda)| > \|p(T)\|. \tag{2.3}$$

But then $p(\lambda) - p(T)$ has a bounded inverse

$$(p(\lambda) - p(T))^{-1} = \sum_{j=0}^{\infty} \frac{p(T)^j}{p(\lambda)^{j+1}} \tag{2.4}$$

which we use to give a local representation for the resolvent.

3 A representation for the resolvent operator

Given any monic polynomial p of degree d

$$p(\lambda) = \lambda^d + a_1\lambda^{d-1} + \cdots + a_d \tag{3.1}$$

we introduce for $j = 0, 1, \ldots, d-1$

$$p_j(\lambda) = \lambda^j + a_1\lambda^{j-1} + \cdots + a_j. \tag{3.2}$$

Then one verifies easily that

$$p(\lambda) - p(T) = q(\lambda, T, p)(\lambda - T) \tag{3.3}$$

where

$$q(\lambda, T, p) = \sum_{j=0}^{d-1} p_{d-1-j}(\lambda)T^j = \sum_{j=0}^{d-1} p_{d-1-j}(T)\lambda^j. \tag{3.4}$$

If now $\lambda \notin V^d(T)$ then by definition there exists $p \in \mathbb{P}_d$ such that (2.3) holds. But then (2.4) and (3.3) give us immediately a simple local representation for the resolvent. We formulate three separate results.

Lemma 3.1. *If $\lambda \notin V^d(T)$, there exists a monic $p \in \mathbb{P}_d$ satisfying (2.3), such that*

$$(\lambda - T)^{-1} = q(\lambda, T, p) \sum_{k=0}^{\infty} p(\lambda)^{-k-1} p(T)^k. \tag{3.5}$$

Take now any open Ω containing $\hat{\sigma}(T)$.

Lemma 3.2. *If Ω is open such that $\hat{\sigma}(T) \subset \Omega$, then there exists d such that $V^d(T) \subset \Omega$.*

Proof. If not, then for each j there exists $\mu_j \in V^j(T)$ such that $\mu_j \notin \Omega$. But the complement of Ω is closed and as $\hat{\sigma}(T)$ is compact there exists $\delta > 0$ such that $dist(\lambda, \hat{\sigma}(T)) \geq \delta$ for all $\lambda \notin \Omega$. However, as $\mu_j \in V^1(T)$ and as $V^1(T)$ is compact, there is a subsequence μ_{j_l} converging to a point in $V(T) = \hat{\sigma}(T)$, violating $dist(\mu_{j_l}, \hat{\sigma}(T)) \geq \delta$. $\square$

Lemma 3.3. *If Ω is open such that $\hat{\sigma}(T) \subset \Omega$, then there exist open sets $U_1, \ldots, U_N$ and monic polynomials $\{q_1, \ldots, q_N\}$ such that the following holds:*

$$\mathbb{C} \setminus \Omega \subset \cup_{j=1}^N U_j, \tag{3.6}$$

$$|q_j(\lambda)| > \lim \|q_j(T)^k\|^{1/k} \text{ for all } \lambda \in U_j, \tag{3.7}$$

and so the series (3.5) converges for $\lambda \in U_j$ with $p = q_j$.

Proof. Let us denote by $r(T)$ the spectral radius of T:

$$r(T) = \lim \|T^k\|^{1/k}.$$

Then clearly for $|\lambda| > r(T)$ we have

$$(\lambda - T)^{-1} = \sum_{k=0}^{\infty} \lambda^{-1-k} T^k$$

which is of the form wanted with $p(\lambda) = \lambda$. Thus all we need to consider is the *compact* set $M = \{\lambda \notin \Omega \ : \ |\lambda| \leq r(T)\}$.

Take any $\mu \in M$. Since $\mu \notin \hat{\sigma}(T)$ there exists, by the definition of polynomial convexity, a polynomial q_μ such that

$$|q_\mu(\mu)| > \max_{z \in \hat{\sigma}(T)} |q_\mu(z)|,$$

which we can assume without loss of generality to be monic. Let U_μ be the open set such that

$$|q_\mu(\lambda)| > \max_{z \in \hat{\sigma}(T)} |q_\mu(z)| \tag{3.8}$$

holds for $\lambda \in U_\mu$. By the spectral radius formula applied to $q_\mu(T)$ we can rewrite (3.8) as

$$|q_\mu(\lambda)| > \lim \|q_\mu(T)^k\|^{1/k} \tag{3.9}$$

and we see that the series in (3.5) converges for $\lambda \in U_\mu$ with $p = q_\mu$. But now the open sets U_μ cover M and as M is compact, it follows that we only need a finite number of such polynomials q_μ. $\square$

4 Green's function for a bounded operator

In order to discuss the convergence speed of the series representation

$$(\lambda - T)^{-1} = q(\lambda, T, p) \sum_{k=0}^{\infty} p(\lambda)^{-k-1} p(T)^k$$

we shall associate a "Green's function" with the operator T, following [11] and [9]. Consider subharmonic functions of the form

$$u_p(\lambda) = \frac{1}{deg(p)} \log \frac{|p(\lambda)|}{\|p(T)\|} \tag{4.1}$$

where p is any monic polynomial. If we now take the supremum of $u_p(\lambda)$ over $p \in \mathbb{P}$, pointwise in λ the limit function vanishes in $\hat{\sigma}(T)$ and is positive or infinite outside. We put for all $\lambda \in \mathbb{C}$

$$g(\lambda, T) = \sup_{p \in \mathbb{P}} u_p(\lambda) = \sup_{p \in \mathbb{P}} \frac{1}{deg(p)} \log \frac{|p(\lambda)|}{\|p(T)\|}, \tag{4.2}$$

and call it the *Green's function for T*. By spectral radius formula one sees that it's values only depend on T thru $\hat{\sigma}(T)$. It does satisfy the mean value property of subharmonic functions but the upper semicontinuity may be lost in taking the supremum. We could do an upper regularization to it in order to get upper semicontinuity, but it does not bring any added value to us here.

Remark 4.1. If the boundary of $\hat{\sigma}(T)$ is regular, then $g(\lambda, T)$ equals the classical harmonic Green's function for the outside of $\hat{\sigma}(T)$.

Remark 4.2. The Green's function $g(\lambda, T)$ is finite for all λ whenever the logarithmic capacity of $\sigma(T)$is positive. On the other hand, if $cap(\sigma(T)) = 0$, then $g(\lambda, T) = \infty$ for all $\lambda \notin \hat{\sigma}(T)$.

5 Convergence of partial sums

Given any monic polynomial p we denote

$$R_m(\lambda, T, p) = q(\lambda, T, p) \sum_{k=0}^{m} p(\lambda)^{-1-k} p(T)^k \tag{5.1}$$

where $q(\lambda, T, p)$ is given in (3.4). Notice that R_m is a rational function in λ of exact degree $d(m+1)$ and a polynomial in T of exact degree $d(m+1)-1$, where d is the degree of p. It is therefore natural to ask for bounds on

$$\|(\lambda - T)^{-1} - R_m(\lambda, T, p)\|$$

in terms of $(m+1)d$.

Take any open Ω containing $\hat{\sigma}(T)$. Then it follows from Lemma 3.3 that there are monic polynomials $q_1, \dots, q_N$ such that for all $\lambda \notin \Omega$

$$f(\lambda) := \max_{j=1,\dots,N} u_{q_j}(\lambda) > 0 \tag{5.2}$$

where u_{q_j} is as in (4.1). It follows, as Ω is open, that

$$\inf_{\lambda \notin \Omega} f(\lambda) > 0,$$

and, since $g(\lambda, T) \geq f$,

$$\eta_\Omega := \inf_{\lambda \notin \Omega} g(\lambda, T) > 0.$$

Recall, that $\eta_\Omega < \infty$ if and only if $cap(\sigma(T)) > 0$. Fix now θ such that

$$e^{-\eta_\Omega} < \theta < 1. \tag{5.3}$$

Proceeding as in the proof of Lemma 3.3 we now conclude from (4.2) and (5.3) the existence of a finite set of monic polynomials $\{q_j\}_1^N$ and corresponding open sets U_j such that for all $\lambda \in U_j$

$$\theta^{d_j}|q_j(\lambda)| > \|q_j(T)\| \tag{5.4}$$

where $d_j = deg(q_j)$. If $q_j(T) = 0$, then T is algebraic, see below, and the series reduces to the exact resolvent. So, assume $\|q_j(T)\| > 0$ for all j. Then, by (5.4) $1/|q_j(\lambda)|$ is bounded and there is a constant C_1 such that for all j with $p = q_j$ and $\lambda \in U_j$

$$\|R_0(\lambda, T, p)\| = \|q(\lambda, T, p)\|/|p(\lambda)| \leq C_1. \tag{5.5}$$

Combining (5.1), (5.4) and (5.5) then gives with $C = \max_{j=1,\dots,N}\{C_1/(1-\theta^{d_j})\}$

$$\|(\lambda - T)^{-1} - R_m(\lambda, T, p)\| \leq C\theta^{d_j(m+1)}.$$

Theorem 5.1. *Let T be a bounded operator in a Banach space X and suppose Ω is an open set containing $\hat{\sigma}(T)$. Then for every θ satisfying (5.3) there exist a constant C, open sets $U_1, \dots, U_N$ covering the complement of Ω and polynomials $q_1, \dots, q_N$ such that, for $\lambda \in U_j$ and for all $m = 0, 1, \dots$ we have*

$$\|(\lambda - T)^{-1} - R_m(\lambda, T, p)\| \leq C\ \theta^{(m+1)deg(p)}, \tag{5.6}$$

where $p = q_j$.

For some subclasses of bounded operators stronger representations than (5.6) are available.

Definition 5.2. *An operator $T \in B(X)$ is called algebraic, $T \in \mathcal{A}$, if there exists a monic q such that $q(T) = 0$;*

it is called almost algebraic, $T \in \mathcal{AA}$, if there exists a complex sequence $\{a_j\}$ such that

$$\|q_j(T)\|^{1/j} \to 0$$

where $q_j(\lambda) = \lambda^j + a_1\lambda^{j-1} + \cdots + a_j$;

we call it polynomially quasinilpotent, $T \in \mathcal{PQN}$, if there exists monic q such that

$$\|q(T)^k\|^{1/k} \to 0;$$

and it is called quasialgebraic, $T \in \mathcal{QA}$, if there exists a sequence of monic polynomials $\{q_j\}$ such that

$$\|q_j(T)\|^{1/deg(q_j)} \to 0.$$

Remark 5.3. All operators in finite dimensional spaces are algebraic, all projections are algebraic (of degree 2) and e.g. the Fourier transform in $L_2(\mathbb{R})$ is algebraic of degree 4. Further, an operator is algebraic if and only if its resolvent is rational. If $T \in \mathcal{A}$, then the resolvent is simply

$$(\lambda - T)^{-1} = R_0(\lambda, T, p)$$

with p the minimal polynomial.

Remark 5.4. All compact operators are almost algebraic and an operator is almost algebraic if and only if its resolvent is meromorphic in $1/\lambda$ for all $\lambda \neq 0$, [11]. If $T \in \mathcal{AA}$, then a *global* representation is still available. In fact, there exists a *characteristic function* χ, entire in $1/\lambda$ and vanishing at the nonzero spectrum, such that

$$(\lambda - T)^{-1} = \frac{1}{\chi(\lambda)} \sum_{j=0}^{\infty} q_j(T)\lambda^{-1-j},$$

see [11]. Here $\chi(\lambda) = 1 + a_1/\lambda + a_2/\lambda^2 + \dots$ and the polynomials q_j are

$$q_j(\lambda) = \lambda^j + a_1\lambda^{j-1} + \dots + a_j.$$

Remark 5.5. Polynomially quasinilpotent operators are clearly exactly those with finite spectrum. So, if T has a finite spectrum, and p vanishes exactly on the spectrum, then $R_m(\lambda, T, p)$ is a very efficient approximation converging superlinearly for every $\lambda \notin \sigma(T)$ and again we have a single representation in the whole resolvent set.

6 Quasialgebraic operators

Since the Green's function becomes infinite for quasialgebraic operators, the coding does not carry information about the convergence other than it becomes superlinear. In order to code speeds on the superlinear scales we need to refine the coding. To that end we put

$$g_k(\lambda, T) = \sup_{p \in \mathbb{P}_k} u_p(\lambda) = \sup_{deg(p) \leq k} \frac{1}{deg(p)} \log \frac{|p(\lambda)|}{\|p(T)\|} \tag{6.1}$$

Lemma 6.1. *For every fixed $\lambda_0 \in \mathbb{C}$ there exists $p_k \in \mathbb{P}_k$ such that*

$$g_k(\lambda_0, T) = u_{p_k}(\lambda_0). \tag{6.2}$$

Proof. Normalize every polynomial considered in $\mathbb{P}_k$ so that its largest coefficient has absolute valued 1. Then the set over which the supremum is taken is compact. But

$$p \mapsto u_p(\lambda_0)$$

is upper semicontinuous and obtains therefore its maximum. The maximum is 0 if and only if $\lambda_0 \in V^k(T)$ and ∞ if and only if T is algebraic of degree at most k (and $\lambda_0 \notin \sigma(T)$). $\square$

Theorem 6.2. *For $T \in B(X)$ and $\lambda_0 \notin V^k(T)$ let p_k be monic and such that (6.2) holds. Then*

$$\|(\lambda_0 - T)^{-1} - R_0(\lambda_0, T, p_k)\| \leq \|R_0(\lambda_0, T, p_k)\| \frac{e^{-kg_k(\lambda_0,T)}}{1 - e^{-kg_k(\lambda_0,T)}}. \tag{6.3}$$

Proof. Put, for short, $R_0 = R_0(\lambda_0, T, p_k)$. Then

$$(\lambda_0 - T)^{-1} - R_0 = R_0 \sum_{j=1}^{\infty} \frac{p_k(T)^j}{p_k(\lambda_0)^j}$$

implies

$$\|(\lambda_0 - T)^{-1} - R_0\| \le \|R_0\| \sum_{j=1}^{\infty} \frac{\|p_k(T)^j\|}{|p_k(\lambda_0)|^j}$$

and since, by arrangement $\|p_k(T)\|/|p_k(\lambda_0)| = e^{-kg_k(\lambda_0,T)}$, the claim follows. □

Remark 6.3. Observe that for $C > 1$ there is some $\delta > 0$ such that the inequality (6.3) holds for all $|\lambda - \lambda_0| < \delta$

$$\|(\lambda - T)^{-1} - R_0(\lambda, T, p_k)\| \le C\|R_0(\lambda, T, p_k)\| \, \frac{e^{-kg_k(\lambda,T)}}{1 - e^{-kg_k(\lambda,T)}}. \tag{6.4}$$

We could then again use compactness as before and conclude that we can approximate the resolvent with $R_0(\lambda, T, p)$ using a finite number of polynomials $p \in \mathbb{P}_k$ with error controlled by $g_k(\lambda, T)$.

Remark 6.4. Notice that as $k \to \infty$

$$g_k(\lambda, T) \to \infty \tag{6.5}$$

if and only if $\lambda \notin \sigma(T)$ and $cap(\sigma(T)) = 0$; i.e. when T is quasialgebraic.

Remark 6.5. Suppose now that we are solving in a Hilbert space H an equation

$$Ax = b$$

with an invertible quasialgebraic $A \in B(H)$. Then Krylov -methods such as GMRES can be used. Theorem 6.2 guarantees the existence of approximations $x_k \in \mathcal{K}_{k+1}(A, b)$ such that

$$\|x_k - x\| \le \|x_k\| \frac{e^{-kg_k(0,A)}}{1 - e^{-kg_k(0,A)}}.$$

In fact

$$x_k - x = -(R_0(0, A, p_k) - A^{-1})b$$

from which the claim follows immediately.

7 Application to Riesz projections

Let K_1 and K_2 be two compact polynomially convex disjoint sets such that

$$\sigma(T) \subset K_1 \cup K_2 \tag{7.1}$$

and for $j = 1, 2$

$$\sigma(T) \cap K_j \neq \emptyset. \tag{7.2}$$

The *Riesz projection* to the invariant subspace associated with K_1 is

$$P = \frac{1}{2\pi i} \int_\gamma (\lambda - T)^{-1} d\lambda \tag{7.3}$$

where γ is a contour surrounding K_1 and leaving K_2 outside. We can assume that γ consists of a finite number of circular arcs and is therefore also of finite length. In fact, since the compact sets are disjoint, there is a $d > 0$ such that $|z_1 - z_2| \geq d$ for $z_j \in K_j$. Take an open cover of K_1 with discs of radius $d/2$ and choose, by compactness, a finite subcover. Then the outer boundary of the cover can be taken as such a contour γ.

As in the proof of Theorem 5.1 we set $\eta_\gamma := \inf_{\lambda\in\gamma} g(\lambda, T)$ and choose any θ satisfying

$$e^{-\eta_\gamma} < \theta < 1. \tag{7.4}$$

We then obtain analogously a partition of γ into finitely many subarcs γ_j and polynomials q_j so that (5.5) holds for $\lambda \in \gamma_j$. Let $d = \max_{j=1,\ldots,N} deg(q_j)$. Then denote:

$$p_{md}(T) = \sum_{j=1}^{N} \frac{1}{2\pi i} \int_{\gamma_j} R_m(\lambda, T, q_j) d\lambda. \tag{7.5}$$

It is clear that p_{md} is a polynomial of degree at most md. As in Theorem 5.1 we have convergence to P in the operator norm.

Proposition 7.1. *If $T \in B(X)$ and K_j are such that (7.1), (7.2) hold, then the polynomials $p_{md}(T)$ in (7.5)satisfy with some C and for all $m = 0, 1, 2, \ldots$*

$$\|P - p_{md}(T)\| \leq C\ \theta^{md}, \tag{7.6}$$

where P is the Riesz projection in (7.3) and θ satisfies (7.4).

Remark 7.2. In this "application" the connection with Green's function becomes familiar. In fact, you could take a holomorphic f, identically 1 in a neighborhood of K_1 and vanishing in some neighborhood of K_2. Then approximate f in $K_1 \cup K_2$ uniformly by polynomials p_n and compare P with $p_n(T)$.

Remark 7.3. With some extra information on the growth of the resolvent as $|\lambda|$ decreases it is possible to give estimates for norms of Riesz projections.

In fact, assume that $(\lambda - T)^{-1}$ is meromorphic for $|\lambda| > r_m(T)$. Then, near any $r > r_m$ there exists some radius ρ such that the norm of the integral

$$\frac{1}{2\pi i}\int_{|\lambda|=\rho}(\lambda - T)^{-1}d\lambda$$

can be controlled. For details, see Theorem 7.5 in [13].

8 Additional remarks

We include some related remarks.

Remark 8.1. Part of the motivation in representing the resolvent in *positive* powers of T comes from the fact that the operator T may be given only as a "black box" $x \mapsto Tx$. As $\mathcal{K}(T,x)$ is invariant for T we may denote by $T_{[x]}$ the restriction of T to $\mathcal{K}(T,x)$, i.e. $T_{[x]} \in B(\mathcal{K}(T,x))$. It follows that

$$\sigma(T_{[x]}) \subset \hat{\sigma}(T) \tag{8.1}$$

and that the set of vectors $x \in X$ for which

$$\hat{\sigma}(T_{[x]}) = \hat{\sigma}(T) \tag{8.2}$$

is of second category in X. The inclusion (8.1) is simple while the second claim follows from local spectral theory where one studies equations of the form

$$(\lambda - T)f(\lambda) = x$$

for a given $x \in X$. If in a neighborhood of λ_0 there exists an analytic $f(\lambda)$ satisfying the equation in that neighborhood, then one says that λ_0 is in the *local resolvent set* $\rho_T(x)$. The *local spectrum of* T at x is then the complement of $\rho_T(x)$ and denoted by $\sigma_T(x)$. If X_0 is the set of vectors x such that $\sigma_T(x) = \sigma(T)$, then X_0 is of second category in X, see [16] or [10]. Let X_1 be the set of vectors for which (8.2) holds. It is clear that we always have $\sigma_T(x) \subset \sigma(T_{[x]})$ so $X_0 \subset X_1$. We shall discuss the computability of local resolvent set and local resolvent in a separate paper.

Remark 8.2. Finding good polynomials to represent the resolvent is computationally demanding. For each fixed λ denote by p_λ a polynomial in $\mathbb{P}_k$ such that

$$g_k(\lambda, T) = \frac{1}{k}\log\frac{|p_\lambda(\lambda)|}{\|p_\lambda(T)\|} \tag{8.3}$$

with $p_\lambda(z) = \sum_{j=0}^k a_j(\lambda)z^j$ normalized e.g. by $\sum_{j=0}^k |a_j(\lambda)|^2 = 1$. For designing good heuristics for finding effective approximations to p_λ results on the dependence of p_λ in λ and in k would be of great interest.

Remark 8.3. Recall that if a compact K is connected, then $\mathbb{C} \setminus \hat{K}$ can be mapped by a conformal map onto outside a disc. The *Faber* polynomials are then obtained from the conformal map and provide approximations to its powers. If K contains the spectrum of an operator in its interior and the degree of the Faber polynomial is large enough, we obtain a representation for the resolvent outside K with just one polynomial.

Remark 8.4. One of the main uses of explicit representations for the the resolvent is the holomorphic functional calculus. For example, one would like to compute e^{tA} using the formula

$$e^{tA} = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} (\lambda - A)^{-1} d\lambda.$$

In the simplest case, suppose that we have a polynomial p such that

$$V_p(A) = \{\lambda \ : \ |p(\lambda)| \leq \|p(A)\|\}$$

is in the open left half plane. Then we can take a contour around it and represent the resolvent along the contour with just one p in $R_m(\lambda, A, p)$. Observe that the evaluation of contour integral reduces to residue calculus at zeros of this *known* polynomial p.

Remark 8.5. As a final remark, the material of this paper complements the discussion in [11] which where partly based on Lectures at ETH in 1992. This paper was written at ETH in 2007.

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