Helsinki University of Technology Institute of Mathematics Research Reports
Espoo 2010 A588

REAL SOLUTIONS TO CONTROL, APPROXIMATION, FACTORIZATION, REPRESENTATION, HANKEL AND TOEPLITZ PROBLEMS

Kalle Mikkola

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

Helsinki University of Technology Institute of Mathematics Research Reports
Espoo 2010 A588

REAL SOLUTIONS TO CONTROL, APPROXIMATION, FACTORIZATION, REPRESENTATION, HANKEL AND TOEPLITZ PROBLEMS

Kalle Mikkola

Aalto University
School of Science and Technology
Department of Mathematics and Systems Analysis

Kalle Mikkola: *Real solutions to control, approximation, factorization, representation, Hankel and Toeplitz problems*; Helsinki University of Technology Institute of Mathematics Research Reports A588 (2010).

Abstract: *Infinite-dimensional control theory often provides complex solutions to various control problems. However, in practical applications one would like to have real solutions. We show that the standard solutions are real given real data. We call a (possibly matrix- or operator-valued) holomorphic function G* real *(real-symmetric) if $G(\bar{z}) = \overline{G(z)}$ for every z. We show that if such a function can be presented as $G = NM^{-1}$, where $N, M \in \mathcal{H}^\infty$, then we have $G = N_R M_R^{-1}$, where $N_R, M_R \in \mathcal{H}^\infty$ are real and weakly right coprime.*

Consequently, if a real function G has a stabilizing compensator (a function K such that $\left[\begin{smallmatrix} I & -K \\ -G & I \end{smallmatrix}\right]^{-1} \in \mathcal{H}^\infty$), then G has a real doubly coprime factorization and a Youla parameterization of all real stabilizing controllers.

If a system of the form $\dot{x} = Ax + Bu,\ y = Cx + Du$ or of the form $x_{n+1} = Ax_n + Bu_n,\ y_n = Cx_n + Du_n$ has real (possibly unbounded, constant) coefficients A, B, C and D, then the system is stabilizable iff it is stabilizable by a real state-feedback operator. This holds for both exponential stabilization and output stabilization. A real stabilizing state-feedback operator is then given by the standard LQR feedback operator, hence the standard (complex) formulae can be used to find this real solution. Analogous results are established for other optimization, factorization, approximation and representation problems too, covering also standard problems on Hankel and Toeplitz operators.

AMS subject classifications: 93D25, 49N10, 93D15, 47B35, 47A68

Keywords: Real-symmetric functions, real coprime factors, real state-feedback operators, real stabilizing controllers, real optimal control, real Hilbert spaces

Correspondence

Kalle Mikkola
Aalto University
Department of Mathematics and Systems Analysis
P.O. Box 11100
00076 Aalto
Finland
kalle.mikkola@iki.fi
`http://math.tkk.fi/people/kalle.mikkola`

Received 2010-06-07

ISBN 978-952-60-3257-3 (print)	ISSN 0784-3143 (print)
ISBN 978-952-60-3258-0 (PDF)	ISSN 1797-5867 (PDF)

Aalto University
School of Science and Technology
Department of Mathematics and Systems Analysis
P.O. Box 11100, FI-00076 Aalto, Finland
email: math@tkk.fi `http://math.tkk.fi/`

Real solutions to control, approximation, factorization, representation, Hankel and Toeplitz problems

Kalle Mikkola

Abstract: Infinite-dimensional control theory often provides complex solutions to various control problems. However, in practical applications one would like to have real solutions. We show that the standard solutions are real given real data.

We call a (possibly matrix- or operator-valued) holomorphic function G *real* (real-symmetric) if $G(\bar{z}) = \overline{G(z)}$ for every z. We show that if such a function can be presented as $G = NM^{-1}$, where $N, M \in \mathcal{H}^\infty$, then we have $G = N_R M_R^{-1}$, where $N_R, M_R \in \mathcal{H}^\infty$ are real and weakly right coprime. Consequently, if a real function G has a stabilizing compensator (a function K such that $\left[\begin{smallmatrix} I & -K \\ -G & I \end{smallmatrix}\right]^{-1} \in \mathcal{H}^\infty$), then G has a real doubly coprime factorization and a Youla parameterization of all real stabilizing controllers.

If a system of the form $\dot{x} = Ax + Bu,\ y = Cx + Du$ or of the form $x_{n+1} = Ax_n + Bu_n,\ y_n = Cx_n + Du_n$ has real (possibly unbounded, constant) coefficients A, B, C and D, then the system is stabilizable iff it is stabilizable by a real state-feedback operator. This holds for both exponential and output stabilization. A real stabilizing state-feedback operator is then the standard LQR feedback operator, hence the standard (complex) formulae can be used to find this real solution. Analogous results are established for other optimization, factorization, approximation and representation problems too, covering also standard problems on Hankel and Toeplitz operators.

AMS class: 93D25, 49N10, 93D15, 47B35, 47A68.

Keywords: Real-symmetric functions, real coprime factors, real state-feedback operators, real stabilizing controllers, real optimal control, real Hilbert spaces.

Contents

1 Introduction

In much of mathematical infinite-dimensional control theory, only complex formulae for the solutions are obtained. This is particularly natural when complex function theory or related tools are applied. However, in practical applications one would usually like to obtain solutions that are real numbers, real sequences, real-symmetric functions—or that are matrices (or operators) having such entries. We show how for many output-feedback, state-feedback and other control problems, the standard methods yield real solutions if the original system or transfer function is real (that is, real-symmetric: $G(\bar{\cdot}) = \overline{G(\cdot)}$). Both state-space and frequency-domain problems are treated, including optimal control, stabilization, factorization, approximation and representation.

We cover weakly coprime and Bézout coprime factorizations, Youla parameterization of stabilizing compensators (for dynamic output feedback), exponential stabilization and output-stabilization by state feedback, the LQR problem and other, possibly indefinite optimal control problems (such as the $\mathcal{H}^\infty$ minimax control), spectral factorization, Nehari, Hartman and Lax–Halmos Theorems, inner-outer factorization etc.

In Section 2 we give the exact definition of "*real*". Then we show that if a real function has a *weakly coprime factorization*, then it has a weakly coprime factorization with real factors. If it has a coprime factorization, then it has a real doubly coprime factorization and the corresponding Youla formula parameterizes all real stabilizing controllers, that is, all real functions K such that $\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \in \mathcal{H}^\infty$). We recall that also the converse holds [Ino88] [Smi89] [Mik07a]: if a function has a stabilizing controller, then it has a coprime factorization.

A related problem, namely the existence of "stable" (that is, $K \in \mathcal{H}^\infty$) real stabilizing compensators, have been studied in, e.g., [MW09], [Wic10] and [Sta92], and Bass stable rank for real-$\mathcal{H}^\infty$ is 2 [MW09].

The real versions of Tolokonnikov's Lemma and of the inner-outer factorization were established in [MS07]. For the Corona Theorem, the symmetrization of any solution yields a solution (i.e., a left inverse). In Section 3 we show that the same symmetrization method applies to the Hartman and Nehari Theorems and that other methods yield real spectral factorization. Also further results on real-symmetric functions are obtained for later use.

Discrete-time systems and state feedback are defined in Section 4: the "next state" equation is $x_{n+1} = Ax_n + Bu_n$ with the initial state x_0 given, u being the input sequence, and $y_n = Cx_n + Du_n$ being the output of the system.

In Section 5 we show that if a real system is *output-stabilizable* by state feedback, then the "LQ-optimal" state-feedback operator is real. This provides a real output-stabilizing state-feedback operator for the system. Moreover, if a real system is power stabilizable, then it is power stabilizable by a real state-feedback operator. On the other hand, the LQ-optimal control always determines a "canonical" weakly coprime factorization of the transfer function; this canonical factorization is then real too. Corresponding proofs are given in Section 6, where analogous "real results" are given also for *indefinite cost functions*.

In Section 7 we show that every real holomorphic function defined on a neighborhood of the origin has a real realization. Using this and the results of Section 5 we prove the results of Section 2.

Above we refer to discrete-time systems, but essentially all results of previous sections hold for *continuous-time systems* too (where $\dot{x}(t) = Ax(t) + Bu(t)$ with $x(0)$ given, and A and B possibly unbounded), as shown in Section 8.

In Section 9 we derive further "real variants" of standard (complex) Hankel and Toeplitz operator results, including the Lax–Halmos Theorem and the $\mathcal{H}^2_{\rm strong}$ inner–outer factorization.

Almost everything presented in this report has been submitted as [Mik10] except Section 9, Lemma 3.3, Lemma 3.1(iii)–(v), the nonseparable case of Theorem 3.6, and some additional details in proofs and elsewhere. Links to some reference articles can be found at www.math.hut.fi/˜kmikkola/research/

Notation. The following notation is defined later in the following order.
Section 2: $\mathtt{U}, \mathtt{X}, \mathtt{Y}, \mathtt{Z}$; $\mathcal{B}(\mathtt{X}, \mathtt{Y})$, $\mathcal{H}^\infty$; $\mathbb{D}$, $\mathbb{T}$, $\mathbb{N}$; $\mathcal{J}$, $\mathcal{K}$; A_{jk}, $\overline{A}$; $\overline{\Omega}$; $\mathtt{U}_\mathbb{R}$; ℓ^2, $\mathrm{L}^2(\mathbb{T}; \mathtt{U})$, $\mathcal{H}^2(\mathbb{D}; \mathtt{U})$; $i = \sqrt{-1}$; A_R, A_I; "*real*", "*real-symmetric*", $\mathcal{H}^\infty_\mathbb{R}$, $\ell^2_\mathbb{R}$; "*proper*", "*right coprime*", "*weakly right coprime*", "*normalized*".
Section 3: f_R, f_I; $\mathrm{L}^\infty_{\rm strong}$;
Lemma 3.3: $\mathtt{U}_\mathbb{R}$, $\operatorname{Re} u$, A_R and A_I generalized to the non-Hilbert (Banach) case.
Section 4: "*system*" $\left(\frac{A \mid B}{C \mid D}\right)$, "*transfer function*" G, "*realization*"; Z*-transform* $\widehat{u}$; "*state-feedback*" F; "*closed-loop system*", N, M; "*output-stable*", "*power-stable*".
Section 5: "*LQR, LQ*", "*Finite Cost Condition*".
Section 6: J, "*cost function* $\mathcal{J}(x_0, u)$", "J*-minimal*"; $\mathscr{C}, \mathscr{D}$; $\mathcal{U}(x_0)$; "J*-optimal*"; $\mathcal{U}_\mathbb{R}(x_0)$, $\operatorname{Re} u$; "J*-optimal cost operator*" $\mathcal{P}$; "J*-optimal state-feedback*".
Section 8: $\mathbb{C}^+$ and the continuous-time terminology.
Section 9: S, S^*, P_+, P_-; "*Hankel operator*", Γ_F; $\mathcal{H}^2_{\rm strong}$; $\mathrm{L}^2_\mathbb{R}(\mathbb{R}; \mathtt{U})$, *"real"*.

2 Coprime factorization and stabilizing compensators

In this section we define "real" and "coprime". Then we show that if a real function G can be written as $G = NM^{-1}$, where $N, M \in \mathcal{H}^\infty$, then the same can be done with N and M real and weakly coprime. If Bézout coprime N and M can be found, then we can simultaneously have them real and Bézout coprime, and they are contained in a real doubly coprime factorization. Moreover, in the latter case we obtain the Youla parameterization of all real stabilizing controllers for the function G. Conversely, if a real function has a stabilizing controller, then it has a real Bézout coprime factorization and hence also a real stabilizing controller.

By $\mathcal{B}(\mathtt{X}, \mathtt{Y})$ we denote the Banach space of bounded linear operators $\mathtt{X} \to \mathtt{Y}$; by $\mathcal{H}^\infty(\mathtt{Z})$ we denote the Banach space of bounded holomorphic functions $\mathbb{D} \to \mathtt{Z}$, where $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ is the unit disc, $\mathtt{X}$ and $\mathtt{Y}$ are Hilbert spaces and $\mathtt{Z}$ is a Banach space. We set $\mathcal{B}(\mathtt{X}) := \mathcal{B}(\mathtt{X}, \mathtt{X})$, $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$, $\mathbb{N} := \{0, 1, 2, \ldots\}$.

In this article $\mathtt{U}$, $\mathtt{X}$ and $\mathtt{Y}$ denote complex Hilbert spaces with fixed orthonormal bases, say $\mathcal{J}$, $\mathcal{E}$ and $\mathcal{K}$, respectively. Then any $u \in \mathtt{U}$ equals $\sum_{j\in\mathcal{J}} u_j j$, where $u_j := \langle u, j\rangle$, and any $A \in \mathcal{B}(\mathtt{U},\mathtt{Y})$ can be identified with the (possibly uncountable) matrix $(A_{jk})_{j\in\mathcal{J},\,k\in\mathcal{K}}$, where $A \mapsto A_{jk} := \langle Aj, k\rangle\ j \in \mathcal{J},\ k \in \mathcal{K}$ is bilinear and continuous $\mathcal{B}(\mathtt{U},\mathtt{Y}) \to \mathbb{C}$ with norm one. Obviously, $(Au)_j = \sum_j A_{jk}u_j$ $(j \in \mathcal{J})$. However, to make it simple, the reader could consider our "input/output" dimensions finite (i.e., $\mathtt{U} = \mathbb{C}^n$, $\mathtt{Y} = \mathbb{C}^m$), as the main results seem to be new even in that setting.

The conjugate $\overline{A}$ is well defined through $\overline{A}_{jk} := \overline{A_{jk}}$ for every j,k. One easily verifies that $\overline{\alpha A + B} = \overline{\alpha}\overline{A} + \overline{B}$, $\overline{AB} = \overline{A}\ \overline{B}$, $\overline{(\overline{A})} = A$, $\overline{A}^* = \overline{A^*}$, $\overline{A^{-1}} = (\overline{A})^{-1}$, $\|\overline{A}\| = \|A\|$, and $(A^*)_{jk} := \overline{A_{kj}}$ for every j,k when $\alpha \in \mathbb{C}$ and A and B are linear operators or vectors of compatible dimensions. Moreover, for any $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued function f we have $f \in \mathcal{H}^\infty$ iff $\overline{f(\overline{\cdot})} \in \mathcal{H}^\infty$. If $\Omega \subset \mathbb{C}$ is a set, then $\overline{\Omega} := \{\overline{z} \mid z \in \Omega\}$ denotes the set of complex conjugates of the elements of Ω.

By $\mathtt{U}_\mathbb{R}$ we denote the real Hilbert space having the same basis $\mathcal{J}$ as $\mathtt{U}$; e.g., $(\mathbb{C}^n)_\mathbb{R} = \mathbb{R}^n$ and $\ell^2(\mathbb{N};\mathbb{C})_\mathbb{R} = \ell^2(\mathbb{N};\mathbb{R})$, which stands for square-summable functions $\mathbb{N} \to \mathbb{R}$. Note that we use the natural bases of $\mathbb{C}^n$ and ℓ^2. For $\mathrm{L}^2(\mathbb{T};\mathtt{U})$ or $\mathcal{H}^2(\mathbb{D};\mathtt{U})$ the functions $z^n j$ serve as the fixed basis, so a function $\sum_n z^n u_n \in \mathrm{L}^2$ is real iff u_n is real (i.e., $u_n \in \mathtt{U}_\mathbb{R}$) for every n.

Obviously, $\mathtt{U} = \mathtt{U}_\mathbb{R} + i\mathtt{U}_\mathbb{R}$, and $\overline{u + iu'} = u - iu'$ for every $u, u' \in \mathtt{U}_\mathbb{R}$. By Lemma 3.2(g) below, $\|u + iu'\|^2 = \|u\|^2 + \|u'\|^2$, so $\mathtt{U} = \mathtt{U}_\mathbb{R} \oplus i\mathtt{U}_\mathbb{R}$.

Moreover, every $A \in \mathcal{B}(\mathtt{U},\mathtt{Y})$ can be written (uniquely) as $A_R + iA_I$, where $A_R := \frac{1}{2}(A + \overline{A})$ and $A_I := -i(A - A_R)$ are real, and then $\overline{A} = A_R - iA_I$. For example, the matrix $A = \left[\begin{smallmatrix} 2 & i \\ -i & 2\end{smallmatrix}\right] = A^* \geq I$ is positive but not real: $A_R = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2\end{smallmatrix}\right]$, $A_I = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right]$.

A basic reference on real operator algebras is [Li03].

A vector or operator A is called real iff its entries are real. A function $\sum_{k=0}^\infty a_k z^k$ is called real iff the coefficients a_k are real. An equivalent definition is given below.

Definition 2.1 *Let $\Omega = \overline{\Omega} \subset \mathbb{C}$ be open. A holomorphic function $f : \Omega \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ is called* real *(or real-symmetric) if*

$$f(z) = \overline{f(\overline{z})} \qquad (z \in \Omega). \tag{1}$$

By $\mathcal{H}^\infty_\mathbb{R}(\mathcal{B}(\mathtt{U},\mathtt{Y}))$ we denote set (real Banach space) of real elements of $\mathcal{H}^\infty(\mathcal{B}(\mathtt{U},\mathtt{Y}))$.

A sequence is called real *if its elements are real. By $\ell^2_\mathbb{R}(\mathbb{N};\mathtt{U})$ we denote set (real Hilbert space) of real elements of $\ell^2(\mathbb{N};\mathtt{U})$, i.e., the set of square-summable real sequences. An operator $A \in \mathcal{B}(\mathtt{U},\mathtt{Y})$ or vector $A \in \mathtt{U}$ is called* real *if $\overline{A} = A$, i.e., if A is real as a constant function.*

An element $f \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U},\mathtt{Y}))$ is real iff its Fourier coefficients are real (by Lemma 3.1 below), or equivalently, iff it is the Z-transform of a real sequence $\mathbb{N} \to \mathcal{B}(\mathtt{U},\mathtt{Y})$. One more equivalent condition is that $f(z)$ is real for real z. Obviously, $f : \Omega \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ is real iff f_{jk} is real (i.e., real-symmetric) for each $j \in \mathcal{J}$, $k \in \mathcal{K}$. In particular, $A \in \mathcal{B}(\mathtt{U},\mathtt{Y})$ is real iff $A_{jk} \in \mathbb{R}$ for each $j \in \mathcal{J}$, $k \in \mathcal{K}$.

The requirement of a fixed basis in the definition of "real" for vectors, matrices or operators is unavoidable—as in the finite-dimensional case too—but very natural. Indeed, any new "real" basis—that is, any "real" coordinate change—preserves the sets of real vectors, matrices and operators and real-symmetric transfer functions. Thus, e.g., any basis of ℓ^2 consisting of real sequences leads to an equivalent definition of "real".

Next we present coprime factorizations.

Definition 2.2 (Coprime) *Let $N \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U},\mathtt{Y}))$ and $M \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U}))$.*
(a) *A function defined and holomorphic on a neighborhood of* 0 *is called* proper.
(b) *We call N and M* right coprime *if $AM - BN \equiv I$ on $\mathbb{D}$ for some $A, B \in \mathcal{H}^\infty$.*
(c) *We call N and M* weakly right coprime[1] *[Smi89] [Mik09b] if $Mf, Nf \in \mathcal{H}^\infty \implies f \in \mathcal{H}^\infty$ for every proper holomorphic $\mathtt{U}$-valued function f.*
(d) *We call N and M* normalized *if $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is inner (i.e., if $\|\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] u_0\| = \|u_0\|$ a.e. on $\mathbb{T}$ for every $u_0 \in \mathtt{U}$).*

Any real function that can be written as NM^{-1} with $N, M \in \mathcal{H}^\infty$ can be written so with N and M weakly right coprime and normalized:

Theorem 2.3 *Let $N \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U},\mathtt{Y}))$, $M \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U}))$, and let $M(0)$ be invertible.*

If the function NM^{-1} is real, then there exist $N_c \in \mathcal{H}^\infty_{\mathbb{R}}(\mathcal{B}(\mathtt{U},\mathtt{Y}))$, $M_c \in \mathcal{H}^\infty_{\mathbb{R}}(\mathcal{B}(\mathtt{U}))$ such that $M_c(0)$ is invertible, $NM^{-1} = N_c M_c^{-1}$ on a neighborhood of 0*, and N_c and M_c are normalized and weakly right coprime.*

If N and M are right coprime, then so are N_c and M_c.

As shown in the proof (Theorems 2.3, 2.4, and 2.5 are proved in Section 7), we can use the standard LQR constructive formulae for N_c and M_c, using the Riccati equation.

If $\dim \mathtt{U} < \infty$, then the N and M in Theorem 2.3 are weakly right coprime iff $\gcd(N, M) = 1$ [Smi89] [Mik08b, Theorem 2.16], i.e., iff all common divisors are units, that is, $M = AX,\ N = BX,\ A, B \in \mathcal{H}^\infty,\ X \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U})) \implies X^{-1} \in \mathcal{H}^\infty$. Further equivalent characterizations of weak coprimeness are given in [Mik09b] and [Mik08b]. Naturally, we may replace 0 by any $\alpha \in \mathbb{D}$ in Theorem 2.3.

Any stabilizable real transfer function is stabilizable by a real compensator:

Theorem 2.4 (Stabilizing compensator) *Let G be a real proper $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued function. If there exists a proper $\mathcal{B}(\mathtt{Y},\mathtt{U})$-valued function K such that $\left[\begin{smallmatrix} I & -K \\ -G & I \end{smallmatrix}\right]^{-1} \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U} \times \mathtt{Y}))$, then there exists a real proper $\mathcal{B}(\mathtt{Y},\mathtt{U})$-valued function K such that $\left[\begin{smallmatrix} I & -K \\ -G & I \end{smallmatrix}\right]^{-1} \in \mathcal{H}^\infty_{\mathbb{R}}(\mathcal{B}(\mathtt{U} \times \mathtt{Y}))$.*

Further details on (internal, or dynamic output-feedback) stabilization are given in, e.g., [Mik07a], [Smi89] and [Vid85].

[1] Equivalence with the standard definition requires coercivity at 0. This difference is redundant in this article, because in applications we have $M(0)$ invertible. Moreover, in the operator-valued case this definition is more useful. Note: when f is a holomorphic function $\Omega \to \mathtt{U}$, we mean by "$f \in \mathcal{H}^\infty$" that $f_{|\Omega\cap\mathbb{D}}$ is the restriction of an element of $\mathcal{H}^\infty(\mathtt{U})$.

Using the above results, we can present the Youla parameterization of all real stabilizing compensators for G.

Theorem 2.5 (Youla parameterization) *Let G be a real proper $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued function. The condition in Theorem 2.4 holds iff $G = NM^{-1}$, where $M(0)$ is invertible in $\mathcal{B}(\mathtt{U})$ and N and M are right coprime. If the condition holds, then N and M can be chosen so that they are real, by Theorem 2.3. Assume that such real N and M exist.*

Then there exist real $X, Y \in \mathcal{H}^\infty$ such that $X(0)$ is invertible and $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]$ is invertible in $\mathcal{H}^\infty(\mathcal{B}(\mathtt{U}\times\mathtt{Y}))$. Moreover, all proper $\mathcal{B}(\mathtt{Y},\mathtt{U})$-valued functions K satisfying $\left[\begin{smallmatrix} I & -K \\ -G & I \end{smallmatrix}\right]^{-1} \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U}\times\mathtt{Y}))$ are given by the Youla parameterization

$$K = (Y + MQ)(X + NQ)^{-1} \tag{2}$$

where $Q \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{Y},\mathtt{U}))$ is such that $(X + NQ)^{-1}$ is proper. The map $Q \mapsto K$ in (2) is one-to-one. The function K is real iff Q is real.

In some engineering applications one might wish to use (real) non-proper controllers [CWW01] [WC97], which are parameterized by (2) without the requirement that $(X + NQ)^{-1}$ is proper [Mik07a, Theorem 1.1 and Section 3].

Remark 2.6 Theorem 2.5 holds even if we remove "$X(0)$ is invertible and", as one observes from the proof. Thus, any real extension of $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ to an invertible element of $\mathcal{H}^\infty$ will do in the theorem. $\triangleleft$

In the matrix-valued case ($\dim \mathtt{U}, \dim \mathtt{Y} < \infty$) it is always possible to have $K \in \mathcal{H}^\infty$ ("stabilization by a stable controller"), but then we cannot require that K is real unless the real poles and zeros of G satisfy the "positive on real zeros" condition (or "parity interlacing condition"), in which case the problem was solved in [Wic10] in the scalar-valued case. Unlike in that problem, in the problems studied in this article the existence of a solution always implies the existence of a real solution.

The domains of M^{-1} and G require some attention in the operator-valued case:

Remark 2.7 (domains of M^{-1} and G) If $\dim \mathtt{U} < \infty$ and $M(0)$ is invertible, then $\det M$ and hence also M is invertible on $\mathbb{D}$ minus some isolated points. If $\dim \mathtt{U} = \infty$, then one has to be particularly careful with the (possibly disconnected) domains of M^{-1}, G and K in Theorem 2.5. One way to solve this problem would be to consider "=" and "$\in$" on sufficiently small neighborhoods of the origin only. However, if G and K are holomorphic on any open and connected $\Omega \subset \mathbb{D}$, then the equations $G = NM^{-1}$ and (2) actually hold on Ω. In particular, then M and $X + NQ$ are invertible on Ω. [Mik07a, Lemma 6.1] $\triangleleft$

There are several explicit formulae for N, M, X and Y in the literature, mostly corresponding to the solutions of Riccati equations corresponding to an arbitrary output-stabilizable realization of G. We refer below to the most general formulae and observe that they become real if G is real and we use, e.g., the realization of Theorem 7.1.

Remark 2.8 (Constructive formulae) Explicit formulae for N, M, X and Y and robust stabilizing compensators are provided in, e.g., [CO06] and [Cur06] (both in continuous time, but results are analogous in discrete time too [CO07]).

All these formulae are given in terms of a realization Σ of G such that Σ and its dual are output-stabilizable. A constructive algorithm for finding such a realization is given in [Mik09b, Remark 5.3]. Moreover, that algorithm and the formulae mentioned above yield real results if the data is real, by Theorems 7.1 and 5.3. $\triangleleft$

3 Real operators

In this section we further elaborate the concept "real" and obtain related results used in the later sections. We also show the existence of real solutions to the Nehari, Hartman, and spectral factorization problems (provided that the data is real and a complex solution exists).

We first present some equivalent characterizations of real-symmetric functions. Recall that a set is *nondiscrete* it contains a non-isolated point.

Lemma 3.1 *Let $f : \Omega \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ be holomorphic and $\Omega = \overline{\Omega} \subset \mathbb{C}$ open and connected. Then the conditions (i)–(v) below are equivalent (and (vi) if $\Omega \cap \mathbb{R} \neq \emptyset$, and (vii) if $0 \in \Omega$).*

(i) *f is real (i.e., $f = \overline{f(\overline{\cdot})}$);*

(ii) *$f = \overline{f(\overline{\cdot})}$ on a nondiscrete subset of Ω;*

(iii) *f' is real and $f(z_0) = \overline{f(\overline{z_0})}$ for some $z_0 \in \Omega$;*

(iv) *$f(r\cdot)$ is real for some (hence every) $r \in \mathbb{R} \setminus \{0\}$;*

(v) *$f(r + \cdot)$ is real for some (hence every) $r \in \mathbb{R}$;*

(vi) *$f(z)$ is real for each $z \in \Omega \cap \mathbb{R}$ (or on a nondiscrete subset of $\Omega \cap \mathbb{R}$).*

(vii) *Every Taylor series (at 0) coefficient $\widehat{f}(n)$ is real ($n \in \mathbb{N}$).*

Proof: (Most of this was presented already in [MS07, Lemma 2.1].)

(i),(ii),(iv),(v): The equivalence of (i), (ii), (iv) and (v) is obvious (e.g., if $f = \overline{f(\overline{\cdot})}$ on a nondiscrete subset of Ω, then $f = \overline{f(\overline{\cdot})}$ on Ω, hence then f is real).

(iii): Obviously, $f'(\overline{z}) = \lim_{\mathbb{R} \ni h \to 0} h^{-1}[f(\overline{z} + h) - f(\overline{z})] = \overline{f'(z)}$ for each $z \in \Omega$ if f is real. The converse follows from the Taylor series at z_0 (or from the fact that $f(z) - f(z_0) = \int_{z_0}^{z} f(\zeta)\, d\zeta$).

(vi): If $\Omega \cap \mathbb{R} \neq \emptyset$, then (vi) is a special case of (ii), as $\overline{f(\overline{\cdot})} = \overline{f}$ on $\Omega \cap \mathbb{R}$.

(vii): Assume that $0 \in \Omega$. If f is real, then so is $\widehat{f}(n)$ for every $n \in \mathbb{N}$, as in Lemma 3.2(b2) below. Conversely, if (vii) holds, then $f(z) = \sum_{n \in \mathbb{N}} \widehat{f}(n) z^n$ is real for each real z. $\square$

Next we record a few more facts on real elements. Here any functions have a Lebesgue-measurable domain $Q \subset \mathbb{C}$ such that $Q = \overline{Q}$ (and the dimensions are assumed to be compatible in (f)).

Lemma 3.2 **(a)** *The functions (or constants)* $f_R := \frac{1}{2}(f + \overline{f(\bar{\cdot})})$ *and* $f_I := (-if)_R$ *are real and* $f = f_R + if_I$ *when* f *is a function (or constant) with values in* $\mathbb{C}$, $\mathtt{U}$ *or* $\mathcal{B}(\mathtt{U},\mathtt{Y})$. *Moreover,* f_R *and* f_I *are unique,* $\overline{f(\bar{\cdot})} = f_R - if_I$, *and* $\overline{f(z)} = f_R(\bar{z}) - if_I(\bar{z})$.

(b1) *If* $f \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U},\mathtt{Y}))$, *then* $f_R(z) = \sum_{n=0}^\infty \widehat{f}(n)_R z^n$ *and* $f_I(z) = \sum_{n=0}^\infty \widehat{f}(n)_I z^n$.

(b2) *If* $f \in \mathrm{L}^1(\mathbb{T};\mathcal{B}(\mathtt{U},\mathtt{Y}))$ *is real-symmetric, then* $\widehat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-in\theta} f(\mathrm{e}^{i\theta})\, d\theta$ *is real-valued.*

(c) *The operations* $f \mapsto f_R$ *and* $f \mapsto f_I$ *are real-linear.*

(d) *If* f *is holomorphic or bounded on* $\mathbb{D}$ *or on a right half-plane, then so are* $\overline{f(\bar{\cdot})}$, f_R *and* f_I.

(e) *Moreover,* f *is real iff* $f = f_R$.

(f) *If* $g = g_R$, *then* $(fg)_R = f_R g$ *and* $(gh)_R = gh_R$.

(g) *If* $u, u' \in \mathtt{U}$ *are real, then* $\|u + iu'\|^2 = \|u\|^2 + \|u'\|^2$.

(h) *We have* $\|f_R\|_p \le \|f\|_p = \|\overline{f(\bar{\cdot})}\|_p$ *for* $f \in \mathrm{L}^p$, $p \in [1,\infty]$, *but this does not hold pointwise; e.g., if* $f(z) := 1 + iz$, *then* $f_R(i) = 1$ *but* $f(i) = 0$.

(i) *If* $A \in \mathcal{B}(\mathtt{U},\mathtt{Y})$ *is real, then* $\|A\| = \sup_{u=u_R\in\mathtt{U},\, \|u\|\le 1} \|Au\|_{\mathtt{Y}}$.

(j) *If* g *is a real one-to-one map of* Ω_1 *onto* Ω_2, *where* $\Omega_1, \Omega_2 \subset \mathbb{C}$, $\Omega_1 = \overline{\Omega_1}$, *then* g^{-1} *is real. Moreover, then a function* $h : \Omega_2 \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ *is real iff* $h \circ g$ *is real.*

From (i) we observe that the natural embedding $\mathcal{B}(\mathtt{U}_\mathbb{R},\mathtt{Y}_\mathbb{R}) \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ defined by $A(u+iv) := Au + iAv$ is a real-linear isometry, so $\mathcal{B}(\mathtt{U}_\mathbb{R},\mathtt{Y}_\mathbb{R})$ can be identified with the set of real elements of $\mathcal{B}(\mathtt{U},\mathtt{Y})$.

Note that $f_R = \operatorname{Re} f$ if f is a constant (operator), e.g., if $f \in \mathcal{B}(\mathtt{U},\mathtt{Y})$, but for $f(z) = z$ we have $f_R = f \ne \operatorname{Re} f$. Moreover, if $f = ig$ for some real g, then $f_R \equiv 0$ even if g is unbounded or nonholomorphic.

Proof of Lemma 3.2 (a)&(c)&(e) Now $f_R(\bar{z}) = \frac{1}{2}[f(\bar{z}) + \overline{f(z)}] = \frac{1}{2}\overline{[f(z) + \overline{f(\bar{z})}]} = \overline{f_R(z)}$. Moreover, $f_I = \frac{1}{2}(-if + i\overline{f(\bar{\cdot})})$, so $f_R + if_I = f$. Now $0 = 0_R + i0_I$, hence $0_R = -i0_I$, so $\overline{0_R(z)} = 0_R(\bar{z}) = -i0_I(\bar{z}) = \overline{i0_I(z)} = \overline{i}0_I(z) = \overline{-0_R(z)}$, hence $0_R(z) = 0 = 0_I(z)$, for every z. Therefore, if $f_R + if_I = g + ih$ for some real g, h, then $f_R - g = 0 = f_I - h$. The maps $f \mapsto f_R, f_I$ are obviously real-linear. The rest is clear and (e) is trivial.

(b1) The subsums form convergent series [HP57, p. 97], so this follows from the uniqueness of f_R, f_I (see (a)).

(b2) $\int_{-\pi}^0 \cdots = \int_0^\pi \mathrm{e}^{in\alpha} f(\mathrm{e}^{-i\alpha})\, d\alpha = \overline{\int_0^\pi \cdots}$, hence $\int_{-\pi}^0 + \int_0^\pi$ is real.

(d) Holomorphicity follows (locally) from (b1) and translation.

(f) We have $f(z)g(z) + \overline{f(\bar{z})g(\bar{z})} = [f(z) + \overline{f(\bar{z})}]g(z)$.

(g) Now $\|u + iu'\|^2 = \sum_k |u_k + iu_k'|^2 = \sum_k |u_k|^2 + \sum_k |iu_k'|^2 = \|u\|^2 + \|u'\|^2$. Thus, $\mathtt{U} = \mathtt{U}_\mathbb{R} \oplus i\mathtt{U}_\mathbb{R}$, where $\mathtt{U}_\mathbb{R}$ is the real Hilbert space $\{u \in \mathtt{U} \bigm| u = u_R\}$.

(h) Obviously, $M := \|f\|_p = \|f(\bar{\cdot})\|_p = \|\overline{f(\bar{\cdot})}\|_p$. Consequently, $\|f_R\|_p \le \frac{1}{2}(M + M) = M$, by Hölder's Inequality.

(i) Set $T := A|_{\mathtt{U}_\mathbb{R}}$. Now A is isomorphic to $A' := \left[\begin{smallmatrix} T & 0 \\ 0 & T \end{smallmatrix}\right] \in \mathcal{B}(\mathtt{U}_\mathbb{R}^2, \mathtt{Y}_\mathbb{R}^2)$, by (g), and $\|A'\| = \max\{\|T\|, \|T\|\} = \|T\|$.

(j) Now $\Omega_2 = \overline{\Omega_2}$, because $g(\bar{z}) = \overline{g(z)}$. Set $F := g^{-1}$. For every $g(z) \in \Omega_2$, we have $F(\overline{g(z)}) = F(g(\bar{z})) = \bar{z} = \overline{F(g(z))}$. Moreover, if h is real, then $h(g(\bar{z})) = h(\overline{g(z)}) = \overline{h(g(z))}$, so $h \circ g$ is real. The converse is analogous, QED. $\square$

Recall that $f \in \mathrm{L}^\infty_{\text{strong}}$ means that $fu \in \mathrm{L}^\infty$ for every u. We set $\|f\|_{\mathrm{L}^\infty_{\text{strong}}} := \sup_{u \in \mathtt{U},\ \|u\| \le 1} \|fu\|_\infty$. In the following lemma we establish some additional results without requiring $\mathtt{U}$ and $\mathtt{Y}$ to be Hilbert spaces; e.g., $\mathtt{U} = \mathcal{H}^\infty$, $\mathtt{U}_\mathbb{R} = \mathcal{H}^\infty_\mathbb{R}$, or $\mathtt{U} = \mathrm{L}^\infty_{\text{strong}}$, $\mathtt{U}_\mathbb{R} = \mathrm{L}^\infty_{\text{strong},\mathbb{R}}$.

Lemma 3.3 (Case: $\mathtt{U}, \mathtt{Y}$ Banach spaces) *Even if we allowed $\mathtt{U}, \mathtt{Y}$ to be arbitrary complex Banach spaces, the following would hold for any real Banach spaces $\mathtt{U}_\mathbb{R}$ and $\mathtt{Y}_\mathbb{R}$ such that $\mathtt{U}_\mathbb{R} + i\mathtt{U}_\mathbb{R} = \mathtt{U}$, $\mathtt{U}_\mathbb{R} \cap i\mathtt{U}_\mathbb{R} = \{0\}$, $\mathtt{Y}_\mathbb{R} + i\mathtt{Y}_\mathbb{R} = \mathtt{Y}$, $\mathtt{Y}_\mathbb{R} \cap i\mathtt{Y}_\mathbb{R} = \{0\}$.*

(a) Then each $u \in \mathtt{U}$ has a unique representation $a + ib$ with $a, b \in \mathtt{U}_\mathbb{R}$, so $\operatorname{Re} u := a$, $\operatorname{Im} u := b$ and $\bar{u} := a - ib$ are well defined, bounded real-linear operations. Obviously, $\overline{\overline{u}} = u$, $\operatorname{Re} u = \frac{1}{2}(u + \overline{u})$, $\operatorname{Im} u = \operatorname{Re}(-iu)$, $\|\operatorname{Im}\| = \|\operatorname{Re}\|$.

(b) We call $A \in \mathcal{B}(\mathtt{U}, \mathtt{Y})$ real *and write $A \in \mathcal{B}_\mathbb{R}(\mathtt{U}, \mathtt{Y})$ if $Au \in \mathtt{Y}_\mathbb{R}$ for every $u \in \mathtt{U}_\mathbb{R}$. The natural mapping $T \mapsto T_C$ given by $T_C(u + iv) := Tu + iTv$ is a real-linear isomorphism of $\mathcal{B}(\mathtt{U}_\mathbb{R}, \mathtt{Y}_\mathbb{R})$ onto $\mathcal{B}_\mathbb{R}(\mathtt{U}, \mathtt{Y})$ and $\mathcal{B}(\mathtt{U}_\mathbb{R}, \mathtt{Y})$ into $\mathcal{B}(\mathtt{U}, \mathtt{Y})$, and $\|T\| \le \|T_C\| \le 2\|\operatorname{Re}\|_{\mathtt{U} \to \mathtt{U}_\mathbb{R}} \|T\|$.*

(c) We have $A_R, A_I \in \mathcal{B}_\mathbb{R}(\mathtt{U}, \mathtt{Y})$, where $A_R u := \operatorname{Re}(A \operatorname{Re} u) + i \operatorname{Re}(A \operatorname{Im} u)$ ($u \in \mathtt{U}$), $A_I := -i(A - A_R)$, for any $A \in \mathcal{B}(\mathtt{U}, \mathtt{Y})$. If $A_R', A_I' \in \mathcal{B}_\mathbb{R}$ and $Au = A_R u + iA_I u$ for every $u \in \mathtt{U}_\mathbb{R}$, then $A_R' = A_R$ and $A_I = A_I'$. The mapping $A \mapsto A_R$ is real-linear and bounded. Finally, $A = A_R$ iff A is real. We set $\overline{A} := A_R - iA_I$.

(d) Lemma 3.1 also holds in this Banach space setting, and so do Lemma 3.2(a)–(f),(j).

(e) Let now $\mathtt{U}, \mathtt{Y}$ be Hilbert spaces. Let $T \in \mathcal{B}(\mathtt{U}_\mathbb{R}, \mathrm{L}^\infty_\mathbb{R}(\mathtt{Y}))$, and set $T_C(u + iv) := Tu + iTv$. This is a real-linear isomorphism $\mathcal{B}(\mathtt{U}_\mathbb{R}, \mathrm{L}^\infty_\mathbb{R}(\mathtt{Y})) \to \mathcal{B}_\mathbb{R}(\mathtt{U}, \mathrm{L}^\infty(\mathtt{Y}))$, and $\|T\| \le \|T_C\| \le 2\|T\|$.

Moreover, there exists $f \in \mathrm{L}^\infty_{\text{strong},\mathbb{R}}(\mathbb{T}; \mathcal{B}(\mathtt{U}, \mathtt{Y}))$ such that $fu = T_C u$ a.e. for every $u \in \mathtt{U}$ and $\|T\| = \sup_\mathbb{T} \|f(z)\|_{\mathtt{U}_\mathbb{R} \to \mathtt{Y}} \le \sup_\mathbb{T} \|f\|_{\mathcal{B}(\mathtt{U},\mathtt{Y})} = \|T_C\|$.

Proof: (a) If $u_n \to 0$ and $\operatorname{Re} u_n \to c \in \mathtt{U}_\mathbb{R}$, as $n \to +\infty$, then $i \operatorname{Im} u_n = u_n - \operatorname{Re} u_n \to 0 - c = -c \in \mathtt{U}_\mathbb{R} \cap i\mathtt{U}_\mathbb{R} = \{0\}$, hence Re is bounded, by the closed-graph theorem.

(b) (This definition of "real" is obviously an extension of that in Definition 2.1.)

Now T_C is bounded, by the closed-graph theorem, because if $u_n + iv_n \to 0$, where $u_n, v_n \in \mathtt{U}_\mathbb{R}$ for every n, then $u_n, v_n \to 0$, because Re, Im are bounded, so then $T_C(u_n + iv_n) \to 0$. Trivially, $\|T\| \le \|T_C\|$, and for $w := u + iv$ we have $\|T_C w\| = \|Tu + iTv\| \le \|T\|\|u\| + \|T\|\|v\| \le 2\|T\|\|\operatorname{Re}\|\|w\|$. (We may have $\|T\| < \|T_C\|$, see (e) below.) Conversely, if $A \in \mathcal{B}_\mathbb{R}(\mathtt{U}, \mathtt{Y})$, then $A_{|\mathtt{U}_\mathbb{R}} \in \mathcal{B}(\mathtt{U}_\mathbb{R}, \mathtt{Y}_\mathbb{R})$.

(c) For $u \in \mathtt{U}_\mathbb{R}$ we have $A_R iu = i \operatorname{Re} Au = iA_R u$, so A_R is complex-linear, hence so is A_I. For $u \in \mathtt{U}_\mathbb{R}$ we have $A_R u = \operatorname{Re} Au \in \mathtt{Y}_\mathbb{R}$ and $A_I u = -i(Au - A_R u) = -i(Au - \operatorname{Re} Au) = -ii \operatorname{Im} Au \in \mathtt{Y}_\mathbb{R}$, hence $A_R, A_I \in \mathcal{B}_\mathbb{R}$. Obviously, $\|A_R\| \le 2\|\operatorname{Re}\|^2\|A\|$, and A_R and A_I are uniquely determined by their restrictions to $\mathtt{U}_\mathbb{R}$. But A is real iff $Au = \operatorname{Re} Au = A_R u$ for every $u \in \mathtt{U}_\mathbb{R}$ (or equivalently, $Au = A_R u$ for every $u \in \mathtt{U}$).

(d) Same proofs apply.

(e) The third sentence is given in (b), because $\|\operatorname{Re}\|_{\mathtt{U}\to\mathtt{U}_\mathbb{R}} \le 1$. Replace X by $X_\mathbb{R}$ (= our $\mathtt{U}_\mathbb{R}$) in formula (10) of the proof of [Mik08a, Theorem 2.5] $\mathbb{R}$ by $\mathbb{T}$ and to obtain $F : \mathbb{T} \to \mathcal{B}(\mathtt{U}_\mathbb{R}, \mathtt{Y})$ such that $\sup_\mathbb{T} \|F\| \le \|T\|$ (obviously, $\sup_\mathbb{T} \|F\| \ge \|T\|$ too) and $Fu = Tu$ a.e. for every $u \in \mathtt{U}_\mathbb{R}$. Set $f(z) := F(z)_C \in \mathcal{B}(\mathtt{U}, \mathtt{Y})$ (see (b)) for every $z \in \mathbb{T}$ to get $f(z)u = (T_C u)(z)$ a.e. for every $u \in \mathtt{U}$, so $f \in \mathrm{L}^\infty_{\text{strong}}$. Now, $\|f(z)\|_{\mathtt{U}_\mathbb{R}\to\mathtt{Y}} = \|F(z)\|$ for every z, so the last claim holds (by Lemma 3.4(d), the inequality may be strict). For $u \in \mathtt{U}_\mathbb{R}$ we have $fu = Tu \in \mathrm{L}^\infty_\mathbb{R}$, hence $f \in \mathrm{L}^\infty_{\text{strong},\mathbb{R}}$ (Lemma 3.4(b)). □

Our Nehari result and some others are based on the following.

Lemma 3.4 ($\mathbf{L^\infty_{strong}}$) *(a) If $f : \mathbb{T} \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ is Bochner-measurable, strongly measurable, L^∞ or $\mathrm{L}^\infty_{\text{strong}}$, then so are $\bar{f}$, $f(\bar{\cdot})$, $\overline{f(\bar{\cdot})}$, f_R and f_I.*

(b) A function $f \in \mathrm{L}^\infty_{\text{strong}}(\mathbb{T}; \mathcal{B}(\mathtt{U}, \mathtt{Y}))$ is real iff fu is real for all $u \in \mathtt{U}_\mathbb{R}$.

(c) Moreover, we have $\|f_R\|_{\mathrm{L}^\infty_{\text{strong}}} \le \|f\|_{\mathrm{L}^\infty_{\text{strong}}}$ for any $f \in \mathrm{L}^\infty_{\text{strong}}$.

(d) However, if $\dim \mathtt{U} \ge 2$ and $\dim \mathtt{Y} \ge 1$, then there exists a real-symmetric $f \in \mathrm{L}^\infty_{\text{strong}}(\mathbb{T}; \mathcal{B}(\mathtt{U}, \mathtt{Y}))$ such that $\|f\|_{\mathrm{L}^\infty_{\text{strong}}} > \sup_{u=u_R\in\mathtt{U},\ \|u\|\le 1} \|fu\|_\infty$.

(Claim (d) can be written as $\|f\|_{\mathcal{B}(\mathtt{U},\mathrm{L}^\infty(\mathtt{Y}))} > \|f\|_{\mathcal{B}(\mathtt{U}_\mathbb{R},\mathrm{L}^\infty_\mathbb{R}(\mathtt{Y}))}$, where $\mathrm{L}^p_\mathbb{R}$ is the real-symmetric subset of L^p. In Corollary 9.1(4.3) it will be shown that $\|f\|_{\mathcal{B}(\mathtt{U},\mathrm{L}^\infty(\mathtt{Y}))} = \|f\|_{\mathcal{B}(\mathrm{L}^2_\mathbb{R}(\mathtt{U}),\mathrm{L}^2_\mathbb{R}(\mathtt{Y}))}$. Claim (b) means, of course, that some function (namely f_R) in the equivalence class of f is real-symmetric if the condition holds.)

Proof: Note: the domain of f could as well be $r\mathbb{T} + s$ or $r + i\mathbb{R}$ for any $r, s \in \mathbb{R}$ (the same proofs apply mutatis mutandis; use Cayley transform for (d)).

(a) The first paragraph is straight-forward. E.g., if, for each $u \in \mathtt{U}$ there exist countably-valued and measurable functions $g_n : \mathbb{T} \to \mathtt{Y}$ ($n \in \mathbb{N}$) such that $g_n \to fu$ a.e. as $n \to +\infty$, then $\bar{g}_n \to \bar{f}\bar{u}$ a.e. Since $\bar{u} \in \mathtt{U}$ was arbitrary, $\bar{f}$ is then strongly measurable. (All operations are well defined: if $fu = gu$ a.e. for each $u \in \mathtt{U}$, then, e.g., $\bar{f}u = \overline{f\bar{u}} = \overline{g\bar{u}} = \bar{g}u$ a.e. for each u.)

(b) If fu is real for all $u \in \mathtt{U}_\mathbb{R}$, then $fu = (fu)_R = f_R u$, by Lemma 3.2(f), for $u \in \mathtt{U}_\mathbb{R}$, so then $f = f_R$. The converse is obvious.

(c) Assume that $\|f\|_{\mathrm{L}^\infty_{\text{strong}}} < \infty$. By [Mik09a, Proposition 2.2], we can redefine f so that $M := \sup_{z\in\mathbb{T}} \|f(z)\| = \|f\|_{\mathrm{L}^\infty_{\text{strong}}}$ (but fu is unchanged a.e., for each $u \in \mathtt{U}$). Now $\|f_R(z)u\| = \frac{1}{2}(\|f(z)u\| + \|\overline{f(\overline{z})}u\|) \le M\|u\|$ for each $u \in \mathtt{U}$, hence $\|f_R\|_{\mathrm{L}^\infty_{\text{strong}}} \le M$, QED. (Alternatively, use Lemma 3.3(e).)

(d) The norm of $f(z) := \begin{bmatrix}1 - z^2 & 1 + z\end{bmatrix}$ is $2 + 4(\operatorname{Im} z)^2$. Set $w := f(i) = \begin{bmatrix}1 + i & 2\end{bmatrix}$. Then $f \in \mathrm{L}^\infty(\mathbb{T}; \mathcal{B}(\mathbb{R}^2, \mathbb{R}))$ and $\|f\|_\infty = |f(i)| = \sqrt{6}$, but $|wu| < |w| \cdot |u|$ for $u \in \mathbb{R}^2$, because $w \notin \mathbb{R}^{2\times 1}$. Therefore, $\sup_{u\in\mathbb{R}^2,\, |u|\le 1} \|fu\|_\infty < \sqrt{6}$. □

Thus, we have proved that the Nehari (or Page) Theorem provides a real solution for real functions.

Corollary 3.5 (Nehari) *If $f \in \mathrm{L}^\infty_{\text{strong}}(\mathbb{T}; \mathcal{B}(\mathtt{U}, \mathtt{Y}))$ is real, then $\min_{g\in\mathcal{H}^\infty} \|f - g\|_{\mathrm{L}^\infty_{\text{strong}}}$ is achieved by a real g.*

Indeed, if g is minimizing, then so is g_R, because $f = f_R$ and $\|f - g_R\| \le \|f - g_R - ig_I\|$, by Lemma 3.4(c), where g_R, g_I are as in Lemma 3.2(a). The fact that a minimizing g exists, is well known [Pag70, Theorem 4] [Pel03, Theorem 2.2 and p. 70] [Mik07c, Corollary 4.5].

However, this "symmetrization" method does not similarly apply to the Adamjan–Arov–Krein problem (as given by, e.g., [Pel03, Theorem 1.1] or [Mik07c, Theorem 4.6]) for $n > 1$, because, e.g., $f(z) = 1/(1 - iz/2)$ has Hankel rank 1 (since $f(z) = \sum_{k=0}^\infty (i/2)^k z^k$), but f_R has Hankel rank 2. We omit the straightforward details.

Nevertheless, [BMS05, Theorems 4.1 and 3.3] yield constructive formulas for real factors and real solutions to the Nehari–Takagi problem if the data A, B, C and G are real (because then so are L_C, L_B, V, Λ, so K is real if(f) we take the parameter Q real).

We observe that also the real version of *Hartman's Theorem* holds. Indeed, if $f \in \mathrm{L}^\infty_{\text{strong}}(\mathbb{T}; \mathcal{B}(\mathtt{U}, \mathtt{Y}))$ has a compact "Hankel operator" Γ_f (see Section 9), then $\Gamma_f = \Gamma_g$ for some continuous $g : \mathbb{T} \to \mathcal{BC}(\mathtt{U}, \mathtt{Y})$, where $\mathcal{BC}$ stands for compact operators, by Hartman's Theorem ([Pel03, p. 74] [Pag70, Sections 4&6] [Mik07c, Theorem 4.7]). Moreover, g_R has the same properties if f is real, because then the coefficients $\widehat{f}(n)$ are real, by Lemma 3.2(b2), and $\widehat{f}(n) = \widehat{g}(n)$ $(n \ge 1)$, by Hartman's Theorem. As $\widehat{g}(n) = \widehat{g_R}(n) + i\widehat{g_I}(n)$ is real, we have $\widehat{g_R}(n) = \widehat{g}(n) = \widehat{f}(n)$ (for $n \ge 1$). By Theorem 3.4(c) and continuity, $\|g_R\|_\infty \le \|g\|_\infty$.

Next we present a standard result on spectral factorization with the additional fact that the factor can be taken real if the original function is real and coercive.

Theorem 3.6 (Spectral factorization) *Let $F \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U}, \mathtt{Y}))$. If $\epsilon > 0$ and $F^*F \ge \epsilon I$ a.e. on $\mathbb{T}$, then there exists $G \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U}))$ such that $G^{-1} \in \mathcal{H}^\infty$ and $F^*F = G^*G$ a.e. on $\mathbb{T}$. If F is real, then we can ensure that G is real too.*

Proof: 1° Separable case: If $F = fg$ is an inner-outer factorization with $g \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U}, \mathtt{W}))$ for some separable Hilbert space $\mathtt{W}$, then $g^*g = F^*F$ a.e. on $\mathbb{T}$, and we know that g is invertible in $\mathcal{H}^\infty$ because of the assumption on F (see, e.g., the proof of [Sta97, Lemma 18], which is based on [RR85]).

Since $g(0)$ is invertible, we have $\dim \mathtt{W} = \dim \mathtt{U} \leq \infty$, so there exists a (unitary) operator $E \in \mathcal{B}(\mathtt{W}, \mathtt{U})$ that maps the fixed basis of $\mathtt{W}$ to that of $\mathtt{U}$. Set $G := Eg \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U}))$ to complete the proof (if F is real, then we can have g (and f) real, by [MS07, Theorem 2.5]; obviously, E is real and hence so is then G).

2° General case: Work analogously to "2°" of the proof of [Mik09a, Theorem 4.3] to obtain the general case from the separable case just proved (using Theorem 9.2 below; it is independent of this). Alternatively, 1° above could be modified using Theorem 9.2. □

However, if $F = i = G$, then $G_R = 0$, so the symmetrization G_R of a solution is not always a solution to $F^*F = G^*G$.

Also many other standard results on Toeplitz and Hankel operators can be reproved for the real case, using the tools developed here, as shown in Section 9.

4 Discrete-time systems

We first recall some details on linear, time-invariant discrete-time systems. See, e.g., [Mik02], [OC04], [Sta05] or [Mik09b] for further details.

A *discrete-time system* on $(\mathtt{U}, \mathtt{X}, \mathtt{Y})$ is a quadruple $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in \mathcal{B}(\mathtt{X} \times \mathtt{U}, \mathtt{X} \times \mathtt{Y})$. For each (square-summable) *input* (or *control*) $u \in \ell^2(\mathbb{N}; \mathtt{U})$ and *initial state* $x_0 \in \mathtt{X}$, we associate the *state* trajectory $x : \mathbb{N} \to \mathtt{X}$ and *output* $y : \mathbb{N} \to \mathtt{Y}$ through

$$\begin{cases} x_{k+1} = Ax_k + Bu_k, \\ y_k = Cx_k + Du_k, \end{cases} \quad k \in \mathbb{N}. \tag{3}$$

The *transfer function* $G := D + C(\cdot^{-1} - A)^{-1}B = D + \cdot C(I - \cdot A)^{-1}B$ of $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is holomorphic $r^{-1}\mathbb{D} \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$, where $r^{-1}\mathbb{D} = \{z \in \mathbb{C} \,\big|\, |z| < r\}$ and $r := r(A)$ is the spectral radius of A. We call $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ a *realization* of G. The *Z-transform* $\widehat{u}$ of $u : \mathbb{N} \to \mathtt{U}$ is defined by $\widehat{u}(z) := \sum_n z^n u_n$ (for those z for which the sum converges absolutely). For $x_0 = 0$, we have $\widehat{y} = G\widehat{u}$ on $\mathbb{D} \cap r^{-1}\mathbb{D}$ for every $u \in \ell^2(\mathbb{N}; \mathtt{U})$, hence the name "transfer function".

State feedback means that, for some *state-feedback operator* $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$, we use the function $u := Fx + u_\circlearrowleft$ as the input, where $u_\circlearrowleft : \mathbb{N} \to \mathtt{U}$ denotes an exogenous input (or disturbance) $u_\circlearrowleft$. Thus, equation (3) together with $u = Fx + u_\circlearrowleft$ defines the "closed-loop system" that maps x_0 and $u_\circlearrowleft$ to x and $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$. The solution is given by (in place of $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$)

$$\left(\begin{array}{c|c} A + BF & B \\ \hline \begin{bmatrix} C + DF \\ F \end{bmatrix} & \begin{bmatrix} D \\ I \end{bmatrix} \end{array}\right), \tag{4}$$

The transfer function of the *closed-loop system* (4) is obviously given by

$$\begin{bmatrix} N(z) \\ M(z) \end{bmatrix} = \begin{bmatrix} D \\ I \end{bmatrix} + \begin{bmatrix} C + DF \\ F \end{bmatrix} (z^{-1} - A - BF)^{-1} B. \tag{5}$$

We conclude from the above that $G = NM^{-1}$. Later we shall see that if F is chosen to be the "LQ-optimal feedback" and $\left(\begin{smallmatrix} A & | & B \\ C & | & D \end{smallmatrix}\right)$ is real, then N and M are real and weakly coprime. The same holds even if we use the standard normalization. This will lead to a proof of Theorem 2.3.

The system (3) is called *output-stable* if $y \in \ell^2$ whenever $x_0 \in \mathtt{X}$ and $u = 0$; *power-stable* if $x \in \ell^2$ whenever $x_0 \in \mathtt{X}$ and $u = 0$. The system (3) is called *output-stabilizable* (resp. *power-stabilizable*) if the system (5) is output-stable (resp. power-stable) for some $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$.

5 LQ-optimal control

We observe here that the "LQ-optimal" state-feedback operator is real if the system is real, and, consequently, any output- or power-stabilizable system can be output- or power-stabilized by a real state-feedback operator. The proofs will be given in Section 6. We assume that $\left(\begin{smallmatrix} A & | & B \\ C & | & D \end{smallmatrix}\right) \in \mathcal{B}(\mathtt{X} \times \mathtt{U}, \mathtt{X} \times \mathtt{Y})$, as above.

The *LQR problem* (Linear Quadratic Regulator problem) means, given an initial state $x_0 \in \mathtt{X}$, finding $u \in \ell^2$ such that the *LQR cost function* $\|y\|_2^2 + \|u\|_2^2$ is minimized. It is probably the most popular control problem in the literature. In this section we shall now see how the solution of this problem is connected to stabilization by (the LQ-optimal) state feedback.

It is well known that if a system can be formally stabilized, then it can be stabilized by state feedback, as stated in Theorem 5.1 below. By formal stabilization we mean the *Finite Cost Condition*:

$$\text{for each } x_0 \in \mathtt{X} \text{ there exists } u \in \ell^2(\mathbb{N}; \mathtt{U}) \text{ such that } y \in \ell^2. \tag{6}$$

If $\left(\begin{smallmatrix} A & | & B \\ C & | & D \end{smallmatrix}\right)$ are real, then, by linearity, an equivalent condition is:

$$\text{for each real } x_0 \in \mathtt{X} \text{ there exists } u \in \ell^2(\mathbb{N}; \mathtt{U}) \text{ such that } y \in \ell^2. \tag{7}$$

We could require the u in (7) to be real-valued, by Theorem 5.1 below. By (3), then x and y become real too.

Theorem 5.1 *Assume the Finite Cost Condition* (7). *Then there exists a unique* $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$ *such that for each* $x_0 \in \mathtt{X}$ *the (state-feedback) input given by* $u_j = F(A + BF)^j x_0$ $(j \in \mathbb{N})$ *strictly minimizes the function* $\|y\|_2^2 + \|u\|_2^2$.

If A*,* B*,* C *and* D *are real, then so is* F*. The functions* N *and* M *in* (5) *are weakly right coprime and* F *is output-stabilizing.*

(Theorems 5.1 and 5.3 will be proved after Lemma 6.4 below, although only F etc. being real is new.)

The Finite Cost Condition is trivially also necessary to make to function $\|y\|_2^2 + \|u\|_2^2$ finite; moreover, it is equivalent to output-stabilizability. The operator F is called the *LQ-optimal state-feedback operator.*

Thus, if a real system is output-stabilizable, then it is output-stabilizable by a real state-feedback operator (namely the LQ-optimal one), which, in addition, makes the closed-loop transfer functions $\mathcal{H}^\infty$ and weakly right coprime.

It will be shown in [Mik07b] that this LQ-optimal operator F is also has the best possible stabilizability properties under many alternative stabilizability assumptions (e.g., that operator might make the state x bounded or even $x_j \to 0$ as $j \to +\infty$, for every initial state x_0 and every external disturbance input $u_\circlearrowleft \in \ell^2(\mathbb{N};\mathtt{U})$ to the closed-loop system (4), whenever achievable by some feedback).

If also the dual system $\left(\begin{array}{c|c} A^* & C^* \\ \hline B^* & D^* \end{array}\right)$ is output-stabilizable, then the functions N and M in Theorem 5.1 are right coprime [CO06].

Theorem 5.1 implies the following (set $C = I$ and $D = 0$ to have $y = x$ and get the claim in parenthesis below).

Corollary 5.2 (stabilizing feedback) *Assume that A, B, C and D are real. If the system is output-stabilizable (resp. power stabilizable), then it is output-stabilizable (resp. power stabilizable) by a real state-feedback operator.*

It is well known that the Finite Cost Condition (6) can be verified by solving the LQR Riccati equation given below and that the solution of this equation determines the LQ-optimal F.

Theorem 5.3 *The system* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *satisfies the Finite Cost Condition* (6) *iff there exists a nonnegative solution $\mathcal{P} \in \mathcal{B}(\mathtt{X})$ of the* LQR Riccati equation

$$A^*\mathcal{P}A - \mathcal{P} + C^*C \tag{8}$$

$$= (C^*D + A^*\mathcal{P}B)(I + D^*D + B^*\mathcal{P}B)^{-1} \tag{9}$$

$$(D^*C + B^*\mathcal{P}A). \tag{10}$$

Assume (6)*. Then there exists a smallest nonnegative solution $\mathcal{P}_{\min}$ and the LQ-optimal state-feedback $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$ is given by*

$$S := I + D^*D + B^*\mathcal{P}_{\min}B, \tag{11}$$

$$F := -S^{-1}(D^*C + B^*\mathcal{P}_{\min}A). \tag{12}$$

Moreover, if A, B, C and D are real, then so are $\mathcal{P}_{\min}$, S and F. Thus, then also $S^{-1/2}$ and the functions $NS^{-1/2}$ and $MS^{-1/2}$ are real; these two functions are also weakly coprime and normalized.

Recall from ((5)) that $G = NM^{-1}$. Also $G = NM^{-1}$ is a weakly coprime factorization but not necessarily normalized.

Most of this section can be considered as well known. Indeed, for some less general settings there are LQR and $\mathcal{H}^\infty$ control results for real Hilbert spaces in the literature. For (continuous-time; cf. Section 8 below) Pritchard–Salamon systems such results are given in [vK93]. The fact that the LQ-optimal

F determines a weakly coprime factorization was established in [Mik09b]. In the case of finite-dimensional systems this has been well known, because, for rational functions, weak coprimeness is equivalent to coprimeness.

6 Optimal control

In this section we shall prove the results of Section 5 in a more general setting, covering also indefinite cost functions in place of the above "LQR cost function" $\|y\|_2^2 + \|u\|_2^2$. The main result of this section is that in real problems the optimal cost operator is real (and so is the optimal state feedback operator etc.).

In this section we assume that operators $\left(\frac{A \mid B}{C \mid D}\right) \in \mathcal{B}(\mathtt{X} \times \mathtt{U}, \mathtt{X} \times \mathtt{Y})$ and a "cost operator" $J = J^* \in \mathcal{B}(\mathtt{Y})$ are given.

We define the *cost function* (to be optimized) by

$$\mathcal{J}(x_0, u) := \langle y, Jy \rangle_{\ell^2} = \sum_{j=0}^{\infty} \langle y_j, Jy_j \rangle_{\mathtt{Y}} \quad (x_0 \in \mathtt{X},\ u \in \ell^2(\mathbb{N}; \mathtt{U})). \tag{13}$$

Recall that the output y is defined by (3). Thus, if $J = I$, we get $\mathcal{J}(x_0, u) = \|y\|_2^2$. By extending C and D (by, e.g., 0 and I and/or I and 0, respectively), we can add copies of u and/or x to the output. Therefore, the cost (13) is very general and covers the LQR cost $\|y\|_2^2 + \|u\|_2^2$ (but (13) may also be indefinite).

Given an initial state $x_0 \in \mathtt{X}$, an input $v \in \ell^2(\mathbb{N}; \mathtt{U})$ is called *J-minimal* for x_0 if $\mathcal{J}(x_0, v) \leq \mathcal{J}(x_0, u)$ for every $u \in \ell^2(\mathbb{N}; \mathtt{U})$.

Denote the maps $x_0 \mapsto y$ and $u \mapsto y$ by $\mathscr{C} := CA^{\cdot}$ and $\mathscr{D}$, respectively. Note that

$$(\mathscr{C}x_0)_k = CA^k x_0 \text{ and } (\mathscr{D}u)_k = \sum_{j=0}^{\infty} CA^j B_{k-j-1} + Du_k \text{ for each } k \in \mathbb{N}. \tag{14}$$

Admissible inputs for x_0 are denoted by $\mathcal{U}(x_0) := \{u \in \ell^2(\mathbb{N}; \mathtt{U}) \mid y \in \ell^2\}$. An input $u \in \mathcal{U}(x_0)$ is called *J-optimal* for x_0 if $\langle y, J\mathscr{D}\eta \rangle_{\ell^2} = 0$ for each $\eta \in \mathcal{U}(0)$.

One can easily verify that a control is J-optimal iff it is a zero of the Fréchet derivative of $\langle y, Jy \rangle_{\ell^2}$ [Mik02, Lemma 8.3.6]. Moreover, if $J \geq 0$, then J-optimal and J-minimal are equivalent, but in minimax problems a J-optimal control can correspond to a saddle point such as the "$\mathcal{H}^\infty$ minimax control" [Sta98] [Mik02].

By $\mathcal{U}_{\mathbb{R}}(x_0)$ we denote the set of real elements of $\mathcal{U}(x_0)$. Given a sequence $u : \mathbb{N} \to \mathtt{U}$, by $\operatorname{Re} u := \frac{1}{2}(u + \overline{u})$ we denote the sequence of real parts of u.

We leave the straightforward proof of the following result to the reader.

Lemma 6.1 *Assume that $\left(\frac{A \mid B}{C \mid D}\right)$ is real. If $x_1, x_2 \in \mathtt{X}$ are real, then $\mathcal{U}(x_1 + ix_2) = \mathcal{U}(x_1) + i\mathcal{U}(x_2) = \mathcal{U}_{\mathbb{R}}(x_1) + i\mathcal{U}_{\mathbb{R}}(x_2)$ (the set $\mathcal{U}(x_1 + ix_2)$ is empty if any of the other four sets is empty). Moreover, if $x_0 \in \mathtt{X}$ is real and $u \in \mathcal{U}(x_0)$, then $\operatorname{Re} u \in \mathcal{U}_{\mathbb{R}}(x_0)$.*

The following operator is very important in applications. It is usually obtained as the (stabilizing) solution of the Riccati equation corresponding to the problem, which is a generalization of (8)–(10).

Definition 6.2 ($\mathcal{P}$) *We call $\mathcal{P} \in \mathcal{B}(\mathtt{X})$ the J-*optimal cost operator *for* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *if, for each $x_0 \in \mathtt{X}$, there exists at least one J-optimal control u with $\mathcal{J}(x_0, u) = \langle x_0, \mathcal{P}x_0 \rangle_{\mathtt{X}}$.*

It follows that $\mathcal{J}(x_0, u) = \langle x_0, \mathcal{P}x_0 \rangle_{\mathtt{X}}$ for every J-optimal control u for x_0 [Mik06][2]; in particular, $\mathcal{P}$ is unique.

We can now prove that $\mathcal{P}$ is necessarily real in real problems.

Theorem 6.3 ($\mathcal{P}$ is real) *Assume that A, B, C, D and J are real. If $x_0 \in \mathtt{X}$ is real and $u \in \ell^2(\mathbb{N}; \mathtt{U})$ is J-optimal for x_0, then $\operatorname{Re} u$ is J-optimal for x_0. Moreover, the J-optimal cost operator, if any, is real.*

Proof: 1° Assume that x_0, u_1 and u_2 are real and $u = u_1 + u_2$. By Lemma 6.1, we have $\mathcal{U}(0) = \{\eta_1 + i\eta_2 \,|\, \eta_1, \eta_2 \in \mathcal{U}_{\mathbb{R}}(0)\}$, so u is J-optimal for x_0 iff $\langle y, J\mathscr{D}\eta \rangle = 0$ for each $\eta \in \mathcal{U}_{\mathbb{R}}(0)$, by linearity. But $y = y_1 + iy_2$, where

$$y_1 := \mathscr{C}x_0 + \mathscr{D}u_1, \quad y_2 := \mathscr{D}u_2. \tag{15}$$

Obviously, y_1 and y_2 are real and $\langle y, J\mathscr{D}\eta \rangle = \langle y_1, J\mathscr{D}\eta \rangle + i\langle y_2, J\mathscr{D}\eta \rangle$, hence u is J-optimal for x_0 iff u_1 is J-optimal for x_0 and u_2 is J-optimal for 0.

2° Let $x_1, x_2 \in \mathtt{X}$ be real. If u_k is real and J-optimal for x_k ($k = 1, 2$), then $\langle x_1, \mathcal{P}x_2 \rangle = \langle \mathscr{C}x_1 + \mathscr{D}u_1, J(\mathscr{C}x_2 + \mathscr{D}u_2) \rangle \in \mathbb{R}$ (expand $\langle x_1 + x_2, \mathcal{P}(x_1 + x_2) \rangle$ to obtain this; use the fact that $u_1 + u_2$ is J-optimal for $x_1 + x_2$). Since x_1 and x_2 were arbitrary, $\mathcal{P}$ is real. $\square$

We call $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$ a *J-optimal state-feedback operator* if the corresponding feedback input $k \mapsto F(A + BF)^k x_0$ (i.e., the input $u = Fx$) is J-optimal for x_0, for every $x_0 \in \mathtt{X}$ (see above (4)). In real problems, F is real:

Lemma 6.4 (F is real) *Assume that A, B, C, D and J are real. If F is a J-optimal state-feedback operator and the J-optimal control for* 0 *is unique, then F is real.*

Proof: Since the J-optimal control for 0 is unique, so is that for any $x_0 \in \mathtt{X}$ (since the difference of two J-optimal controls for x_0 is J-optimal for 0). Let a real $x_0 \in \mathtt{X}$ be given. Then $u := F(A + BF)^{\cdot} x_0$ satisfies $u = \operatorname{Re} u$, by uniqueness and Theorem 6.3, hence u is real, hence $u_0 = Fx_0$ is real. Since x_0 was arbitrary, F is real. $\square$

Proof of Theorems 5.1 and 5.3 This follows from [Mik09b, Theorem 1.2 & Proposition 3.1] except that the realness of $\mathcal{P}$ and F and the uniqueness of F are from Theorem 6.3 and Lemma 6.4 (with $\left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} D \\ I \end{smallmatrix}\right]$ and $\mathtt{Y} \times \mathtt{U}$ in place of J, C, D and $\mathtt{Y}$, respectively); by (11), also S is real; by (5) also N and M are real (also $S^{-1/2}$ is real, by [Chr02, Lemma A.6.7], because $S \geq 0$ and S is real). $\square$

[2] Actually, [Mik06] treats the continuous-time case but the proof is analogous. Even if the J-optimal control were non-unique, the corresponding cost is always unique [Mik06, Lemma 3.5].

7 Proofs for Section 2

In this section we prove the results of Section 2.

Typical feedback stabilization problems are solvable only for transfer functions that can be factorized as NM^{-1}, where $N, M \in \mathcal{H}^\infty$. Many equivalent characterizations of this "factorizability" are given in [Mik09b, Theorem 1.2].

Here we record the fact that every real "factorizable" function is the transfer function of some real output-stabilizable realization (also the converse holds).

Theorem 7.1 (realization) *If G is a real proper $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued function and $G = NM^{-1}$ for some $N, M \in \mathcal{H}^\infty$ such that $M(0)$ is invertible, then the shift realization $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ of G in [Mik09b, Theorem 5.2] is real and output-stabilizable.*

We omit the straightforward proof. The equations $G(z) = N(z)M(z)^{-1}$ and $G(z) = D + C(z^{-1} - A)^{-1}B$ are to hold near the origin. By the realization being real we mean that A, B, C and D are real.

Proof of Theorem 2.3 By Theorem 7.1, NM^{-1} has a real output-stabilizable realization. Theorem 5.3 provides a real normalized "weakly coprime factorization" $N_\mathrm{c}M_\mathrm{c}^{-1}$ of NM^{-1}. The last claim follows from [Mik09b, Theorem 1.1]. □

Proof of Theorems 2.5 and 2.4 1° Without the words "real", Theorem 2.5 is contained in [Mik07a, Theorem 1.1].

2° Assume G is real and as in the theorem. The coprime N and M can be taken real, by Theorem 2.3, and so can Y and X, by [MS07]; assume that they are real. Moreover, as in the proof of [Mik07a, Lemma A.5], we can choose the real $X, Y \in \mathcal{H}^\infty$ so that $X(0) = I$ and $Y(0) = 0$.

3° Because $\left[\begin{smallmatrix} M & Y+MV \\ N & X+NV \end{smallmatrix}\right] = \left[\begin{smallmatrix} M & X \\ N & Y \end{smallmatrix}\right]\left[\begin{smallmatrix} I & V \\ 0 & I \end{smallmatrix}\right]$, we observe that $Y+MV$ and $X+NV$ are real iff V is real.

4° Let $V \in \mathcal{H}^\infty$ be such that the $K = Y_1X_1^{-1}$ in (2) is real, where $X_1 := X+NV$ and $Y_1 := Y+MV$. By [Sta05, Theorem 8.5.7], X_1 and Y_1 are coprime. Now $V = V_R + iV_I$, where $V_R, V_I \in \mathcal{H}^\infty$ are real. Moreover,

$$Y + MV_R + iMV_I = Y_1 = KX_1 = KX + KNV_R + iKNV_I. \tag{16}$$

Therefore, $(M - KN)V_I = 0$, hence $V_I = 0$, because $(M - KN) = M^{-1}(I - KNM^{-1}) = M^{-1}(I - KG)$ has a proper inverse [Mik07a, equation (1)]. Thus, V is real.

5° Conversely, if V is real, then so is K, by 3°, so Theorem 2.5 holds.

6° Take $V = 0$ to observe that Theorem 2.4 holds. □

8 Continuous time results

In this section we prove that the analogies of almost all results of previous sections hold for continuous-time systems too, such as well-posed linear systems

(Salamon–Weiss systems). In particular, the unit disc $\mathbb{D}$ is replaced by the right half-plane $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ and equation (3) is replaced by $\dot{x} = Ax + Bu$, $y = Cx + Du$, $x(0) = x_0$, where A, B and C may be moderately unbounded and D not necessarily well defined. It is often easier to describe the system as $\left[\begin{smallmatrix} \mathscr{A} & \mathscr{B}\tau \\ \mathscr{C} & \mathscr{D} \end{smallmatrix}\right] : \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ with the requirements that the system is linear and time-invariant and maps $x_0 \in \mathtt{X}$, $u \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathtt{U})$ boundedly to $x(t) \in \mathtt{X}$, $y \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathtt{Y})$ for some (hence any) $t > 0$. Further details can be found in, e.g., [SW02] [Sta05] [Mik06] [Mik08b, Section 5] [Mik02] [WC97].

Most results of Section 2 are obtained for $\mathbb{C}^+$ merely by Cayley transforming, as stated in Remark 8.1(b) below. The standard form of "proper" can also be obtained (see (c) below).

In [Mik06], it was shown that formal output stabilizability (i.e., the Finite Cost Condition) implies stabilizability by well-posed state feedback, by showing that the LQ-optimal state-feedback is well-posed (for parabolic systems this was already known). If the system is real, then the LQ-optimal state-feedback is real too, so any real output-stabilizable (resp., exponentially stabilizable) system is stabilized by well-posed real state feedback (by (e) and (f) below). In the proofs we use the tools developed above, and the same tools can be used to obtain "real" forms of many other standard results too.

Remark 8.1 **(a)** *A Laplace-transformable function $f : [0,\infty) \to \mathtt{Z}$ is (essentially) real-valued iff its Laplace-transform $\widehat{f}(z) = \int_0^\infty \mathrm{e}^{-tz} f(t)\,dt$ is real-symmetric.*

(b) *Let $r > 0$. The results of Sections 2–3 (except Lemma 3.1(vii) and Lemma 3.2(b1)&(b2)) also hold with $\mathbb{C}^+$, $\mathbb{R}$ and r in place of $\mathbb{D}$, $\mathbb{T}$ and 0, respectively (in the domains of functions, hence in the definition of $\mathcal{H}^\infty$, "proper", "coprime" etc.)*

(c) *The above result (b) also holds if* "proper" *is redefined as "defined on some right half-plane" (i.e., on $\{\operatorname{Re} z > \omega\}$ for some $\omega \in \mathbb{R}$) except that in Theorem 2.5 it is not known whether X^{-1} can always be taken proper (it can be if, e.g., $\lim_{\operatorname{Re} z \to +\infty} G(z)$ exists).*

(d) *Lemmata 6.1 and 6.4 and Theorem 6.3 also hold if we replace $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ by a linear map $[\ \mathscr{C} \mid \mathscr{D}\ ] : (x_0, u) \mapsto y$ and F (in Lemma 6.4) by any map $\mathscr{F}_\circlearrowleft$ such that $\mathscr{F}_\circlearrowleft(x_0)$ is J-optimal for each $x_0 \in \mathtt{X}$.*

(e) Real version of [Mik06]. *Assume that the map $[\ \mathscr{C} \mid \mathscr{D}\ ]$ of [Mik06] is real and that the* Finite Cost Condition *holds i.e., for each $x_0 \in \mathtt{X}$ there exists $u \in \mathrm{L}^2([0,\infty);\mathtt{U})$ such that $\mathscr{C}x_0 + \mathscr{D}u \in \mathrm{L}^2$ (we can assume x_0 to be real and require u to be real, cf. (7)).*

Then there exists a real LQ-optimal state-feedback pair $[\ \mathscr{F}_\circlearrowleft \mid \mathscr{G}_\circlearrowleft\ ]$ such that (in [Mik06]) the corresponding N and M are real, normalized and weakly coprime, $[\ \mathscr{F} \mid \mathscr{G}\ ]$ are real and $S = I$.

(f) *If $[\ \mathscr{A} \mid \mathscr{B}\ ]$ of [Mik06] are real and the system is exponentially stabilizable, then the system is exponentially stabilizable by a real state-feedback pair.*

(It obvious that a real state-feedback pair corresponds to a real state-feedback operator, as defined in, e.g., [Sta05], [Mik02] and [Mik08b].)

Proof: (a) This is straight-forward (use the Laplace inversion formula for "if" [HP57, Theorem 6.3.2]). Recall that "Laplace-transformable" means that $\mathrm{e}^{-r\cdot} f \in \mathrm{L}^1$ for some $r \in \mathbb{R}$.

Note: Similarly, a function $f \in \mathrm{L}^2(\mathbb{R};\mathtt{U})$ is essentially real-valued iff its Plancherel transform $\widehat{f} : i\mathbb{R} \to \mathtt{U}$ is essentially real-symmetric. For test functions this follows from the above by continuity, for others by density of test functions in L^2 (and by the pointwise convergence a.e. of a subsequence).

(b) The Cayley transform $\phi(z) := (r-z)/(r+z)$ maps $\mathbb{C}^+$ one-to-one and onto $\mathbb{D}$, and $\phi(r) = 0$. It preserves real-symmetricity, by Lemma 3.2(j). Therefore, Theorem 3.6 and the results of Section 2 follow and those of Section 3 arise from same proofs, mutatis mutandis.

(c) This follows from (b) and [Mik08b, Theorem 3.1(b)] (and if N, M, M^{-1} are $\mathcal{H}^\infty$ over $\{\operatorname{Re} z > \omega\}$ for some $\omega \geq 0$, then all a "weakly coprime factorization" in the r-sense are "weakly coprime factorizations" in the half-plane sense too, and vice versa, for any $r > \omega$).

Assume then that $D := \lim_{\operatorname{Re} z \to +\infty} G(z)$ exists. Then, for $F := G - D$ there exists $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right] = \left[\begin{smallmatrix} T & -U \\ -S & R \end{smallmatrix}\right]^{-1} \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U} \times \mathtt{Y}))$ such that $F = NM^{-1} = R^{-1}S$ and that X^{-1} and T^{-1} are proper, by [Mik07a, Theorem 7.4].

Now $\left[\begin{smallmatrix} M_G & Y_G \\ N_G & X_G \end{smallmatrix}\right] := \left[\begin{smallmatrix} I & 0 \\ D & I \end{smallmatrix}\right]\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]\left[\begin{smallmatrix} I & -TY \\ 0 & I \end{smallmatrix}\right] \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U} \times \mathtt{Y}))$ is obviously invertible, $N_G M_G^{-1} = (N + DM)M^{-1} = F + D = G$, and $MT - YS = I$, i.e., $I - MT = YS$. Consequently, $X_G - X = DY - NTY - DMTY = -NTY + D(I - MT)Y = -NTY + DYSY = -FMTY + DYRF$, so $\|X_G(z) - X(z)\| \leq \gamma \|F(z)\| \to 0$, as $\operatorname{Re} z \to +\infty$. Therefore, also X_G^{-1} is uniformly bounded for $\operatorname{Re} z$ big enough.

(d) This is obvious from the proofs. Note that the other two components of a well-posed linear system, namely $\mathscr{A}$ and $\mathscr{B}$, need not be real and they do not affect J, $\mathcal{P}$, S, N, M etc.

(e) Now $G := \widehat{\mathscr{D}}$ has a normalized, weakly coprime factorization $N_1 M_1^{-1}$, by [Mik06, Corollary 5.1]. By (b) and Theorem 2.3, G also has a real, normalized, weakly coprime factorization NM^{-1}. By [Mik09b, Theorem 1.1] normalized weakly coprime factorizations are unique modulo a unitary operator, so [Mik06, Lemmata A.5 & 4.4] yield another LQ-optimal pair corresponding to the factorization NM^{-1}.

Also $\mathscr{N}$ and $\mathscr{M}$ obviously are real (i.e., they map real-valued functions to real-valued functions, by (a); or equivalently, $\mathscr{N}$ and $\mathscr{M}$ are real as elements of $\mathcal{B}(\mathrm{L}^2)$, where the basis of L^2 consists of real-valued functions). By (the proof of) [Mik06, Lemmata 4.4], $S = I$. By [Mik06, (2.7)], $[\ \mathscr{F} \mid \mathscr{G}\ ]$ are real.

(f) This follows from (e) and the proof of [Mik06, Corollary 5.4]. (So Corollary 5.2 holds also in continuous-time setting.) $\square$

In Remark 8.1(c), the assumption that $\lim_{\operatorname{Re} z \to +\infty} G(z)$ exists can be replaced by a more general assumption, but a necessary assumption for a proper real stabilizing compensator K to exist is the so-called "parity interlacing condition" (or "positive on real zeros") [Sta92] [Wic10] on *some* right half-plane. For scalar-valued $G = NM^{-1}$, $N, M \in \mathcal{H}^\infty$, this means that M must have the same sign at every zero of N on some right half-axis $\{z > R\}$.

9 Hankel and Toeplitz operators

In Section 3 we presented "real" variants of some standard (complex) Hankel and Toeplitz operator results including the Nehari Theorem and spectral factorization. In this section we present many more "real" variants, also of most standard results extended in [Mik07c] to general Hilbert spaces as well as the operator diagonalization method of [Mik07c].

First we recall some notation from [Mik07c], again in the discrete-time setting, the functions being defined on the unit circle or disc. By S we denote the shift $f \mapsto zf$, i.e., $(Sf)(z) = zf(z)$. Similarly, $(S^*f)(z) = z^{-1}f(z)$. By

$$P_+ : \sum_{k=-\infty}^{\infty} z^k x_k \mapsto \sum_{k=0}^{\infty} z^k x_k$$

we denote the orthogonal projection $\mathrm{L}^2 \to \mathcal{H}^2$, and we set $P_- := I - P_+$.

The operators $\Gamma \in \mathcal{B}(\mathcal{H}^2_-(\mathcal{X}), \mathcal{H}^2(\mathcal{Y}))$ that satisfy $P_+S^*\Gamma = \Gamma S^*$ are called *Hankel operators*. (In the literature, Hankel operators are often defined in an equivalent way $\mathrm{L}^2 \to \mathcal{H}^2_-$, with z^{-1} in place of z.)

The *Hankel operator* $\Gamma_F \in \mathcal{B}(\mathcal{H}^2_-(\mathcal{X}), \mathcal{H}^2(\mathcal{Y}))$ *of* a function $F \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathcal{X}, \mathcal{Y})$ is defined by $\Gamma_F := P_+M_FP_-$.

Finally, a (necessarily holomorphic) function $F : \mathbb{D} \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ is $\mathcal{H}^2_{\mathrm{strong}}$ if $Fu \in \mathcal{H}^2$ for every $u \in \mathtt{U}$. We set $\|F\|_{\mathcal{H}^2_{\mathrm{strong}}} := \sup_{\|u\|\le 1} \|Fu\|_{\mathcal{H}^2}$.

Corollary 9.1 *In this corollary, the spaces $\mathcal{X}$ and $\mathcal{Y}$ are assumed to be real Hilbert spaces. The numbering of items below refers to corresponding complex results in [Mik07c]. The following claims are true.*

(4.3) *Every operator $T \in \mathcal{B}(\mathrm{L}^2_{\mathbb{R}}(\mathcal{X}+i\mathcal{X}), \mathrm{L}^2_{\mathbb{R}}(\mathcal{Y}+i\mathcal{Y}))$ that satisfies $ST = TS$ equals $M_F : f \mapsto Ff$ for some $F \in \mathrm{L}^\infty_{\mathrm{strong},\mathbb{R}}(\mathcal{X}+i\mathcal{X}, \mathcal{Y}+i\mathcal{Y})$. Moreover,*

$$\|M_F\|_{\mathcal{B}(\mathrm{L}^2_{\mathbb{R}}(\mathcal{X}+i\mathcal{X}),\mathrm{L}^2_{\mathbb{R}}(\mathcal{Y}+i\mathcal{Y}))} = \|M_F\|_{\mathcal{B}(\mathrm{L}^2(\mathcal{X}+i\mathcal{X}),\mathrm{L}^2(\mathcal{Y}+i\mathcal{Y}))} = \|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$$

for every $F \in \mathrm{L}^\infty_{\mathrm{strong},\mathbb{R}}(\mathcal{X}+i\mathcal{X}, \mathcal{Y}+i\mathcal{Y})$.

(4.4) *Every Hankel operator $\Gamma : \mathcal{H}_{-,\mathbb{R}}(\mathcal{X}+i\mathcal{X}) \to \mathcal{H}^2_{\mathbb{R}}(\mathcal{Y}+i\mathcal{Y})$ equals Γ_F for some $F \in \mathrm{L}^\infty_{\mathrm{strong},\mathbb{R}}(\mathcal{X}+i\mathcal{X}, \mathcal{Y}+i\mathcal{Y})$. Moreover, this F can be chosen so that*

$$\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \|\Gamma\| = \|\Gamma\|_{\mathcal{H}^2_-(\mathcal{X}+i\mathcal{X})\to\mathcal{H}^2(\mathcal{Y}+i\mathcal{Y})}.$$

(4.15 "Lax–Halmos") *Every closed, real-symmetric, shift-invariant subspace $\mathcal{M}$ of $\mathcal{H}^2_{\mathbb{R}}(\mathcal{X}+i\mathcal{X})$ satisfies $\mathcal{M} = M_F[\mathcal{H}^2_{\mathbb{R}}(\mathcal{X}_0+i\mathcal{X}_0)]$ for some real subspace $\mathcal{X}_0$ of $\mathcal{X}$ and some inner $F \in \mathcal{H}^\infty_{\mathbb{R}}(\mathcal{X}_0+i\mathcal{X}_0, \mathcal{X}+i\mathcal{X})$.*

(4.17 "Inner–Outer Factorization") *Every $F \in \mathcal{H}^2_{\mathrm{strong},\mathbb{R}}(\mathcal{X}+i\mathcal{X}, \mathcal{Y}+i\mathcal{Y})$ can be expressed as $F = F_iF_o$ with F_i, F_o real-symmetric, $F_o \in \mathcal{H}^2_{\mathrm{strong},\mathbb{R}}(\mathcal{X}+i\mathcal{X}, \mathcal{Y}_0+i\mathcal{Y}_0)$ outer and $F_i \in \mathcal{H}^\infty_{\mathbb{R}}(\mathcal{Y}_0+i\mathcal{Y}_0, \mathcal{Y}+i\mathcal{Y})$ inner, for some closed real subspace $\mathcal{Y}_0$ of $\mathcal{Y}$. Moreover,*

$$\|F_o\|_{\mathcal{H}^2_{\mathrm{strong}}} = \|F\|_{\mathcal{H}^2_{\mathrm{strong}}}, \quad \|F_o\|_{\mathcal{H}^\infty} = \|F\|_{\mathcal{H}^\infty} \le \infty, \quad \textit{and} \quad \dim\mathcal{Y}_0 \le \dim\mathcal{X}.$$

This is contained in Theorem 9.2 below. The case of 4.17 with $F \in \mathcal{H}^\infty$ was already established in [MS07]. The above results share the (discrete-time) notation of [Mik07c, Section 4], so the above L^2 spaces are defined over the unit circle $\mathbb{T}$, and $\mathrm{L}^2_{\mathbb{R}}$ refers to the subset of (essentially) real-symmetric elements, as in Definition 2.1 .

Before presenting the remaining results, in a more messy form, we list some observations for the continuous-time input-output maps or transfer functions (that is essentially: Toeplitz operators or their symbols), as the theorem also contains continuous-time variants (as in [Mik07c, Section 5]) of the above and other results. In that setting, functions are defined on $\mathbb{R}$ instead of $\mathbb{T}$ and the transfer functions are defined on the right half-plane instead of $\mathbb{D}$.

Note first that, for any complex Hilbert space $\mathtt{U}$, any orthonormal basis of (the real Hilbert space) $\mathtt{U}_{\mathbb{R}}$ is an orthonormal basis of $\mathtt{U}$. Conversely, if W is a real Hilbert space, then $W + iW$ with natural operations is a complex Hilbert space. (Here $i(w + iw') := -w' + iw$. The inner product is given by $(w + iw', v + iv') := (w, v) + (w', v') + i(w', v) - i(w, v')$.)

For $\mathrm{L}^2(\mathbb{R};\mathtt{U})$ we fix some basis that consists of real-valued functions. Therefore, an element f of $\mathrm{L}^2(\mathbb{R};\mathtt{U})$ is *real* (i.e., $f \in \mathrm{L}^2_{\mathbb{R}}(\mathbb{R};\mathtt{U})$) iff it is essentially real-valued (i.e., $f \in \mathrm{L}^2(\mathbb{R};\mathtt{U}_{\mathbb{R}})$). Definition 2.1, an operator $A : \mathrm{L}^2(\mathbb{R};\mathtt{U}) \to \mathrm{L}^2(\mathbb{R};\mathtt{Y})$ is *real* iff it maps real elements to real elements, or equivalently, some real basis into $\mathrm{L}^2(\mathbb{R};\mathtt{Y}_{\mathbb{R}})$.

The set of real operators $\mathrm{L}^2(\mathbb{R};\mathtt{U}) \to \mathrm{L}^2(\mathbb{R};\mathtt{Y})$ can be identified with the set of operators $\mathrm{L}^2(\mathbb{R};\mathtt{U}_{\mathbb{R}}) \to \mathrm{L}^2(\mathbb{R};\mathtt{Y}_{\mathbb{R}})$, by Lemma 3.3(b).

A bounded, linear, time-invariant (i.e., translation invariant) operator $A : \mathrm{L}^2(\mathbb{R};\mathtt{U}) \to \mathrm{L}^2(\mathbb{R};\mathtt{Y})$ is real iff its symbol (or transfer function, as in [Mik07c, Theorem 5.2 & Proposition 5.3]) is real-symmetric. This can be easily deduced from Remark 8.1(a) or from the corresponding claim on the Plancherel Transform, depending on whether one treats $\mathcal{H}^\infty$ transfer functions on the right half-plane or $\mathrm{L}^\infty_{\text{strong}}$ symbols on the imaginary axis. Analogous claims obviously hold for shift-invariant operators on ℓ^2 too.

Now we are ready to present the "real variants" of most results of [Mik07c]. Note that in Sections 4 of [Mik07c] and [Mik09a], L^2 refers to functions on $\mathbb{T}$ ("discrete time") and in Sections 5 to functions on $\mathbb{R}$ ("continuous time").

Theorem 9.2 (Real variants of [Mik09a] & [Mik07c])
Omitting Theorems 4.6, 4.10, 5.6 and 5.10 of [Mik07c] and Theorems 4.1 and 5.4 of[Mik09a], each of the Lemmata, Propositions and Theorems of Sections 4–5 of [Mik09a] and [Mik07c] also holds in its "real" form, where the external Banach and Hilbert spaces (including $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$) are assumed to be real, $\ell^p(Q;\mathbb{C})$ is replaced by $\ell^p(Q;\mathbb{R})$ for any set Q, and $\mathcal{P}(\mathbb{C})$ by $\mathcal{P}(\mathbb{R})$, and $\mathcal{C}(\mathcal{X},\mathcal{Y})$ is replaced by the real Banach space $\mathcal{C}_{\mathbb{R}}(\mathcal{X}+i\mathcal{X},\mathcal{Y}+i\mathcal{Y})$ consisting of the real-symmetric elements of $\mathcal{C}(\mathcal{X}+i\mathcal{X},\mathcal{Y}+i\mathcal{Y})$ (with the norm of the latter), and $\mathcal{C}(\mathcal{X})$ by $\mathcal{C}_{\mathbb{R}}(\mathcal{X}+i\mathcal{X})$; similarly for L^p, $\mathrm{L}^p_{\text{strong}}$, $\mathcal{H}$, $\mathcal{H}^p$, $\mathcal{H}^p_{\text{strong}}$, $\mathcal{H}^p_-$, or $\mathcal{C} \circ \mathcal{BC}$ in place of $\mathcal{C}$, for any Hilbert spaces $\mathcal{X},\mathcal{Y}$ and $1 \le p \le \infty$ (and P_X is replaced by P_{X+iX}, similarly for $P_Y, \tilde{P}_X, \tilde{P}_Y$ etc.); however, in Section 5, all L^2 spaces

must remain unchanged.

Theorem 3.1(a1)–(a3) (or (a1)–(a2) of [Mik09a]) hold for real Hilbert spaces too. Moreover, in (the original, complex) Theorem 3.2 of [Mik09a] and Theorem 3.2(a)–(e),(g) of [Mik07c], if F is real (and X_0 and Y_0 in (g) are real), we can have all pairs (X, Y) of the form $(V + iV, W + iW)$, where, V and W are real. Finally, Lemmata A.1-3,8-10 and B.1-2 of [Mik07c] and A.1-4 and B.1 of [Mik09a] hold for real Hilbert and Banach spaces too (use, e.g., $\mathcal{H}^2_{\mathbb{R}}(\mathcal{X} + i\mathcal{X})$ in place of $\mathcal{H}^2(\mathcal{X})$ in A.1).

(Also real variants of many other results can be established. See Theorem 3.3(e) for the "real variant" of Proposition 2.2 of [Mik09a] and [Mik07c].)

Proof: In most cases, these results follow fairly easily from the complex ones (often with the use of our Lemma 3.3) or from their proofs. Nehari, Hartman and spectral factorization theorems were already handled in Section 3 above to some extent. We present below the least trivial parts of the proofs, referring to [Mik07c], which essentially contains all results of [Mik09a].

Theorem 4.3: Note from Lemma 3.3(b) that $\mathcal{B}(\mathrm{L}^2_{\mathbb{R}}(\mathcal{X} + i\mathcal{X}), \mathrm{L}^2_{\mathbb{R}}(\mathcal{Y} + i\mathcal{Y})) = \mathcal{B}_{\mathbb{R}}(\mathrm{L}^2(\mathcal{X}+i\mathcal{X}), \mathrm{L}^2(\mathcal{Y}+i\mathcal{Y}))$ Define T_C as in Lemma 3.3(b). Obviously, $ST = TS$ iff $ST_C = T_C S$. But $Ff = M_F f \in \mathrm{L}^2_{\mathbb{R}}(\mathcal{Y}+i\mathcal{Y})$ for every $f \in \mathrm{L}^2_{\mathbb{R}}$, hence for every constant $f \in \mathtt{X}$. Consequently, $F = F_R$, by Lemma 3.4(b). By Lemma 3.2(i), $\|T\| = \|T_C\|$.

Theorem 4.4: Obviously, a Hankel operator is real iff its Hankel matrix is real, i.e., iff $\widehat{F}(k)$ is real for $k \geq 1$, so F_R will do, by Lemma 3.2(a) ($\|\Gamma\|$ remains unaltered, by Lemma 3.2(i); by Lemma 3.4(c), we have $\|F_R\| \leq \|F\|$).

Corollary 4.5: this is Corollary 3.5 ($\|\Gamma_F\|$ remains unaltered, by Lemma 3.2(i)).

Theorem 4.7: Now $F = F_R$, so Γ_F is compact on $\mathcal{H}^2_-$ iff on $\mathcal{H}^2_{-,\mathbb{R}}$ (because the product of compact sets is compact), But (iii) and (iv) are satisfied by G_R ($\|\Gamma_F\|$ remains unaltered, by Lemma 3.2(i); by 3.4, $\|G\|$ does not increase). We get X and Y from the separably-valued functions in the original proof.

Theorems 4.8, 4.9: modify the proof using [MS07] (in 4.8, note that $\|P_-FP_-(f+ig)\|_2^2 = \|P_-FP_-f\|_2^2 + \|P_-FP_-ig\|_2^2$ for real-symmetric F, f, g, by Lemma 3.2(g)). Results 4.11–4.13, 4.16: obvious (from the old results). Theorem 4.17: old proof.

Theorem 4.14: The old proof applies ($T \in \mathcal{B}(\mathcal{H}^2_{\mathbb{R}}, \mathcal{H}^2_{\mathbb{R}})$ implies $T \in \mathcal{B}_{\mathbb{R}}(\mathcal{H}^2, \mathcal{H}^2)$, by Lemma 3.3(b)). (N.B., if $G \in \mathcal{H}^\infty_{\mathbb{R}}$ is any complex left divisor of F, i.e., $F = GK$ for some $K \in \mathcal{H}^\infty_{\mathbb{R}}$, then $GK_R = F_R = F$, by Lemma 3.2(f), so then G is a real left divisor of F, too.)

Theorem 4.15: This follows from the proof of [MS07, Theorem 2.5] (with $\mathcal{M} + i\mathcal{M}$ in place of $f\mathcal{H}^2(X)$ and $\mathcal{M}$ in place of $f\mathcal{H}^2_{\mathbb{R}}$) complemented by the last five lines of the proof of Theorem 4.15, mutatis mutandis.

The results of Section 5 can be proved in a similar way. As noted before the statement of Theorem 9.2 above, we can identify $\mathrm{TI}(\mathcal{X}, \mathcal{Y})$ with $\mathrm{TI}_{\mathbb{R}}(\mathcal{X} + i\mathcal{X}, \mathcal{Y} + i\mathcal{Y})$, by Lemma 3.3(b). By Lemma 3.2(i), here the two norms coincide. □

Recall that in the subset $L^\infty_{\text{strong},\mathbb{R}}$ we use the L^∞_{strong} norm, not that of Lemma 3.4(d).

Acknowledgments. This work was written with the support of Magnus Ehrnrooth Foundation.

References

[BMS05] Joseph A. Ball, Kalle M. Mikkola, and Amol J. Sasane, *State-space formulas for the Nehari-Takagi problem for nonexponentially stable infinite-dimensional systems*, SIAM J. Control Optim. **44** (2005), no. 2, 531–563 (electronic).

[Chr02] Ole Christensen, *An introduction to frames and Riesz bases*, Birkhäuser, Boston, 2002.

[CO06] Ruth F. Curtain and Mark R. Opmeer, *Normalized doubly coprime factorizations for infinite-dimensional linear systems*, Math. Control Signals Systems **18** (2006), no. 1, 1–31.

[CO07] ———, *The Nehari problem for discrete-time infinite-dimensional systems*, Proceedings of the 2007 European Control Conference (ECC 2007) (Kos, Greece), July 2007.

[Cur06] Ruth F. Curtain, *Robustly stabilizing controllers with respect to left-coprime factor perturbations for infinite-dimensional linear systems*, Systems Control Lett. **55** (2006), no. 7, 509–517.

[CWW01] Ruth F. Curtain, George Weiss, and Martin Weiss, *Stabilization of irrational transfer functions by controllers with internal loop*, Systems, approximation, singular integral operators, and related topics, Oper. Theory Adv. Appl., vol. 129, Birkhäuser, Basel, 2001, pp. 179–207.

[HP57] Einar Hille and Ralph S. Phillips, *Functional analysis and semi-groups*, revised ed., AMS, Providence, 1957.

[Ino88] Yujiro Inouye, *Parametrization of compensators for linear systems with transfer functions of bounded type*, Technical report 88-01, Faculty of Eng. Sci., Osaka Univ., Osaka, Japan, 1988.

[Li03] Bingren Li, *Real operator algebras*, World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[Mik02] Kalle M. Mikkola, *Infinite-dimensional linear systems, optimal control and algebraic Riccati equations*, Doctoral dissertation, technical report A452, Institute of Mathematics, Helsinki University of Technology, Espoo, Finland, 2002, `www.math.hut.fi/~kmikkola/research/` .

[Mik06] ———, *State-feedback stabilization of well-posed linear systems*, Integral Equations Operator Theory **55** (2006), no. 2, 249–271.

[Mik07a] ———, *Coprime factorization and dynamic stabilization of transfer functions*, SIAM J. Control Optim. **45** (2007), no. 2, 1988–2010.

[Mik07b] ———, *Different types of stabilizability and detectability of well-posed linear systems*, manuscript, 2007.

[Mik07c] ———, *Hankel and Toeplitz operators on nonseparable Hilbert spaces: further results*, Technical Report A528, Institute of Mathematics, Helsinki University of Technology, Espoo, Finland, 2007, `www.math.hut.fi/reports/` .

[Mik08a] ———, *Fourier multipliers for L^2 functions with values in nonseparable Hilbert spaces and operator-valued H^p boundary functions*, Ann. Acad. Sci. Fenn. Math. **33** (2008), no. 1, 1–10.

[Mik08b] ———, *Weakly coprime factorization and continuous-time systems*, IMA J. Math. Control Inform. **25** (2008), no. 4, 515–546.

[Mik09a] ———, *Hankel and Toeplitz operators on nonseparable Hilbert spaces*, Ann. Acad. Sci. Fenn. Math. **34** (2009), no. 1, 109–129.

[Mik09b] ———, *Weakly coprime factorization and state-feedback stabilization of discrete-time systems*, Math. Control Signals Systems **20** (2009), no. 4, 321–350 pp.

[Mik10] ———, *Real solutions to control, approximation and factorization problems*, submitted, 2010.

[MS07] Kalle M. Mikkola and Amol J. Sasane, *Tolokonnikov's Lemma for real H^∞ and the real disc algebra*, Complex Anal. Oper. Theory **1** (2007), no. 3, 439–446.

[MW09] Raymond Mortini and Brett D. Wick, *The bass and topological stable ranks of $H^\infty_R(\mathbb{D})$ and $A_R(\mathbb{D})$*, J. Reine Angew. Math. **636** (2009), 1–17.

[OC04] Mark R. Opmeer and Ruth F. Curtain, *Linear quadratic Gaussian balancing for discrete-time infinite-dimensional linear systems*, SIAM J. Control Optim. **43** (2004), no. 4, 1196–1221.

[Pag70] Lavon B. Page, *Bounded and compact vectorial Hankel operators*, Trans. Amer. Math. Soc. **150** (1970), 529–539.

[Pel03] Vladimir V. Peller, *Hankel operators and their applications*, Springer Monographs in Mathematics, Springer, New York, 2003.

[RR85] Marvin Rosenblum and James Rovnyak, *Hardy classes and operator theory*, Oxford University Press, New York, Oxford, 1985.

[Smi89] Malcolm C. Smith, *On stabilization and the existence of coprime factorizations*, IEEE Trans. Automat. Control **34** (1989), no. 9, 1005–1007.

[Sta92] Olof J. Staffans, *Stabilization of a distributed system with a stable compensator*, Math. Control Signals Systems **5** (1992), no. 1, 1–22.

[Sta97] ______, *Quadratic optimal control of stable well-posed linear systems*, Trans. Amer. Math. Soc. **349** (1997), 3679–3715.

[Sta98] ______, *On the distributed stable full information H^∞ minimax problem*, Internat. J. Robust Nonlinear Control **8** (1998), 1255–1305.

[Sta05] ______, *Well-Posed Linear Systems*, Encyclopedia Math. Appl., vol. 103, Cambridge University Press, Cambridge, 2005.

[SW02] Olof J. Staffans and George Weiss, *Transfer functions of regular linear systems. Part II: the system operator and the Lax-Phillips semigroup*, Trans. Amer. Math. Soc. **354** (2002), no. 8, 3229–3262.

[Vid85] Mathukumalli Vidyasagar, *Control system synthesis: a factorization approach*, MIT Press Ser. Signal Process. Optim. Control, 7, MIT Press, Cambridge, MA, 1985.

[vK93] Bert van Keulen, *H_∞-control for distributed parameter systems: A state space approach*, Birkhäuser Verlag, Basel Boston Berlin, 1993.

[WC97] George Weiss and Ruth F. Curtain, *Dynamic stabilization of regular linear systems*, IEEE Trans. Autom. Control **42** (1997), 4–21.

[Wic10] Brett D. Wick, *Stabilization in $H_R^\infty(\mathbb{D})$*, Publ. Mat. **54** (2010), no. 1, 25–52.

(continued from the back cover)

A582 Juho Könnö, Rolf Stenberg
Analysis of H(div)-conforming finite elements for the Brinkman problem
January 2010

A581 Wolfgang Desch, Stig-Olof Londen
An L_p-theory for stochastic integral equations
November 2009

A580 Juho Könnö, Dominik Schötzau, Rolf Stenberg
Mixed finite element methods for problems with Robin boundary conditions
November 2009

A579 Lasse Leskelä, Falk Unger
Stability of a spatial polling system with greedy myopic service
September 2009

A578 Jarno Talponen
Special symmetries of Banach spaces isomorphic to Hilbert spaces
September 2009

A577 Fernando Rambla-Barreno, Jarno Talponen
Uniformly convex-transitive function spaces
September 2009

A576 S. Ponnusamy, Antti Rasila
On zeros and boundary behavior of bounded harmonic functions
August 2009

A575 Harri Hakula, Antti Rasila, Matti Vuorinen
On moduli of rings and quadrilaterals: algorithms and experiments
August 2009

A574 Lasse Leskelä, Philippe Robert, Florian Simatos
Stability properties of linear file-sharing networks
July 2009

HELSINKI UNIVERSITY OF TECHNOLOGY INSTITUTE OF MATHEMATICS
RESEARCH REPORTS

The reports are available at *http://math.tkk.fi/reports/* .
The list of reports is continued inside the back cover.

A587 Antti Hannukainen, Rolf Stenberg, Martin Vohralík
A unified framework for a posteriori error estimation for the Stokes problem
May 2010

A586 Kui Du, Olavi Nevanlinna
Minimal residual methods for solving a class of R-linear systems of equations
May 2010

A585 Daniel Aalto
Boundedness of maximal operators and oscillation of functions in metric measure spaces
March 2010

A584 Tapio Helin
Discretization and Bayesian modeling in inverse problems and imaging
February 2010

A583 Wolfgang Desch, Stig-Olof Londen
Semilinear stochastic integral equations in L_p
December 2009

ISBN 978-952-60-3257-3 (print)
ISBN 978-952-60-3258-0 (PDF)
ISSN 0784-3143 (print)
ISSN 1797-5867 (PDF)