Department of Mathematics and Systems Analysis

Multicentric representation and von Neumann spectral sets

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Aalto University

SCIENCE + TECHNOLOGY

RESEARCH REPORT

Aalto University publication series
SCIENCE + TECHNOLOGY 11/2011

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AMS Mathematics Subject Classification 2010: 47A25, 30B99, 47A60
Manuscript received 2011-05-25

ISBN 978-952-60-4163-6 (pdf)
ISBN 978-952-60-4162-9 (printed)
ISSN-L 1799-4896
ISSN 1799-490X (pdf)
ISSN 1799-4896 (printed)

Aalto Print
Helsinki 2011

Finland

The publication can be read at http://math.tkk.fi/reports/

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Abstract

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Name of the publication
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Publisher School of Science

Unit Department of Mathematics and Systems Analysis

Series Aalto University publication series SCIENCE + TECHNOLOGY 11/2011

Field of research Mathematics

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ISBN (printed) 978-952-60-4162-9 **ISBN (pdf)** 978-952-60-4163-6

ISSN-L 1799-4896 **ISSN (printed)** 1799-4896 **ISSN (pdf)** 1799-490X

Location of publisher Espoo **Location of printing** Espoo **Year** 2011

Pages iv + 12 **The publication can be read at** http://math.tkk.fi/reports/

Multicentric representation and von Neumann spectral sets

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Abstract

We show how multicentric representation of functions provides a simple way to generalize the von Neumann result that the unit disc is a spectral set for contractions in Hilbert spaces. In particular the sets need not be connected and the results can be applied to bounding Riesz spectral projections.

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AMS 47A25, 30B99, 47A60

1 Multicentric representation of holomorphic functions

1.1 Motivation

As power series converge in discs it has been natural to treat holomorphic functions in discs, use conformal maps to help in dealing with more complicated domains, and if need be, to use local variables or charts to provide accurate local representations of functions. However, in concrete computations this often leads to path dependent approaches and and then much of the basic power of function theory such as residue calculus is lost.

A natural source for the need of a practical calculus in complicated geometries comes from the holomorphic functional calculus. Recall that any compact nonempty set can be the spectrum of an element in some Banach algebra.

Let $\mathcal{A}$ be a unital Banach algebra and assume given an element $a \in \mathcal{A}$ and a scalar function f which is holomorphic in a neighborhood of the spectrum of a. In order to have a systematic way to express $f(a) \in \mathcal{A}$ and estimate its size effectively, we, rather than trying to spread out the function on a suitable covering surface, take a very opposite angle: *we try to pack the information many times over on as small a neighborhood of the origin as possible*.

In [5] we provided an algorithm which, for a given $a \in \mathcal{A}$, produces a sequence of monic polynomials p with distinct roots such that the sets

$$V_p(a) = \{z \in \mathbb{C} \,:\, |p(z)| \leq \|p(a)\|\} \tag{1.1}$$

squeeze around the polynomially convex hull of the spectrum of $a \in \mathcal{A}$. In particular, if f is holomorphic in some neighborhood of the polynomially convex hull of the spectrum, then there exists a p such that f is holomorphic in a neighborhood of $V_p(a)$ as well. Assuming such a polynomial we next outline the multicentric representation of f which it induces. More details can be found in [6].

Let $p \in \mathbb{P}_d$ be a monic polynomial of degree d with simple roots λ_j and let $\delta_k \in \mathbb{P}_{d-1}$ denote for $k = 1, \ldots, d$ the polynomials which take the value 1 at λ_k and vanish at the other roots. We then form the *multicentric representation* of f with respect to centers λ_k:

$$f(z) = \sum_{k=1}^{d} \delta_k(z) f_k(w), \quad where \quad w = p(z). \tag{1.2}$$

So, rather than using several local variables or charts we use two *global* variables z and w, associate to each center λ_k a function f_k in the variable w and finally combine them together using the basis polynomials in the original variable z. One may view the construction as a combination of *Lagrange interpolation* and *Jacobi series*. In [6] we have discussed the computation of the functions f_k and shown in particular that their Taylor series can be computed in a natural recursive fashion if the derivatives of the original function f are available at the local centers.

The representation allows an obvious avenue for analysis, estimation and computation in complicated sets. One just treats the functions f_k in discs $|w| \leq R$ and combines the estimates for f in the sets satisfying $|p(z)| \leq R$.

1.2 Basic estimate

In this paper we demonstrate this approach by generalizing a well known result of von Neumann on contractions in Hilbert spaces. In order to do this we need to have an estimate of the following form

$$\sup_{|w| \leq R} |f_k(w)| \leq C(R) \sup_{|p(z)| \leq R} |f(z)|. \tag{1.3}$$

Such an estimate would then imply that the sets $V_p(A)$ are K*-spectral sets* with some K.

In order to state the estimate we need some notation. Let γ_R denote the lemniscate

$$\gamma_R = \{z \in \mathbb{C} \,:\, |p(z)| = R\}.$$

For small R the lemniscate consists of d separate circular curves, for large R it reduces to just one circular curve. In general the lemniscate is smooth except if it contains a critical point, where the derivative of p vanishes. Thus there are at most $d - 1$ such exceptional values R. Let $s(R)$ denote the distance from γ_R to the set of critical points.

Theorem 1.1. *If p is a monic polynomial of degree d with distinct roots, then there exists a constant C such that if f is holomorphic for $|p(z)| \leq R$, then the functions f_k in (1.2) are holomorphic for $|w| \leq R$ and if γ_R does not contain any critical points of p the estimate (1.3) holds with some $C(R)$ satisfying*

$$C(R) \leq 1 + \frac{C}{s(R)^{d-1}}. \tag{1.4}$$

Remark 1.2. If $C(R)$ denotes the smallest constant such that (1.3) holds for all f then $C(R) \to 1$ as $R \to 0$ or $R \to \infty$. Generically the critical points are simple and then the constant is proportional to $1/s(R)$ but we include an example where the behavior is of the form $1/s(R)^{d-1}$.

We postpone the proof but first apply this to spectral set theory.

2 Applications to spectral sets

2.1 K-spectral sets using the von Neumann Theorem

Let $\mathcal{A}$ be a unital Banach algebra. We recall the definitions related to this topic.

Definition 2.1. *A closed set $\Sigma \subset \mathbb{C}$ is a spectral set for $a \in \mathcal{A}$, if for all rational functions R with poles off Σ there holds*

$$\|R(a)\| \leq \sup_{z\in\Sigma} |R(z)|. \tag{2.1}$$

If the equation holds in the form

$$\|R(a)\| \leq K \sup_{z\in\Sigma} |R(z)|,$$

with a fixed K, then Σ is called a K-spectral set.

The topic began with a fundamental result by von Neumann for contractions in Hilbert spaces.

Theorem 2.2. *(von Neumann, 1951 [7])*
If $A \in B(H)$, and $\|A\| \leq 1$, then the closed unit disc is a spectral set for A.

This can clearly be reformulated also as follows:

$$\|f(A)\| \leq \sup_{|z|\leq\|A\|} |f(z)| \tag{2.2}$$

provided f is holomorphic in $|z| \leq \|A\|$.

We formulate our results for holomorphic functions rather than for polynomials or rational functions as we consider sets which may consist of several components. Here is the main result of this paper.

Theorem 2.3. *Suppose we are given a monic $p \in \mathbb{P}_d$ with distinct roots and a bounded operator $A \in B(H)$ in a Hilbert space H. Let $R \geq 0$ satisfy $\|p(A)\| \leq R$ and be such that the lemniscate γ_R contains no critical points of p. Then for all f which are holomorphic for $|p(z)| \leq R$ there holds*

$$\|f(A)\| \leq K \sup_{|p(z)|\leq R} |f(z)|, \tag{2.3}$$

where the constant K satisfies

$$K \leq C(R) \sum_{k=1}^{d} \|\delta_k(A)\|, \tag{2.4}$$

with $C(R)$ as in Theorem 1.1.

Proof. The claim follows immediately from Theorem 1.1 and from the von Neumann Theorem 2.2. In fact, denoting $B = p(A)$ we have from (2.2)

$$\|f_k(B)\| \leq \sup_{|w|\leq R} |f_k(w)|$$

and so by Theorem 1.1

$$\|f_k(B)\| \leq C(R) \sup_{|p(z)|\leq R} |f(z)|.$$

Then the result follows from

$$f(A) = \sum_{k=1}^{d} \delta_k(A) f_k(B).$$

□

2.2 Application to the Riesz spectral projections

A simple but useful application of the previous result is obtained as follows. Suppose γ_R consists of several components and is free from critical points. Then one can define f to be identically 1 in some open neighborhood of some of the components and to vanish in a neighborhood of all the others. If $A \in B(H)$ is such that $\|p(A)\| \leq R$, then the resulting operator is simply the *Riesz spectral projection* to the invariant subspace w.r.t. the part of the spectrum where f equals 1.

The following example shows that the constant $C(R)$ of Theorem 1.1 has to blow up near the critical lemniscates, and that the worst behavior in (1.4) may happen.

Example 2.4. Let $\varepsilon > 0$ be small. Consider the matrix

$$A(\varepsilon) = \begin{pmatrix} \varepsilon & 1 \\ 0 & -\varepsilon \end{pmatrix}, \tag{2.5}$$

with spectrum $\sigma(A(\varepsilon)) = \{\varepsilon, -\varepsilon\}$. Let $p(\lambda) = \lambda^2 - 1$ so that we have one critical point at the origin. Put $R = 1 - \varepsilon^2$ so that the spectrum lies on the boundary of the lemniscate $|p(z)| = \|p(A(\varepsilon))\| = 1 - \varepsilon^2$. Let f be 1 on the right open half plane and 0 on the left open half plane, so in particular it is holomorphic inside and in a neighborhood of the lemniscate. Then $f(A(\varepsilon)) = P(\varepsilon)$ is well defined and equals the *Riesz spectral projection* onto the direction of the eigenvector wrt the eigenvalue ε. In fact, the resolvent satisfies

$$(\lambda I - A(\varepsilon))^{-1} = \frac{1}{\lambda^2 - \varepsilon^2} \begin{pmatrix} \lambda + \varepsilon & 1 \\ 0 & \lambda - \varepsilon \end{pmatrix}$$

The Riesz projector can be obtained as the residue at ε:

$$P(\varepsilon) = \begin{pmatrix} 1 & 1/2\varepsilon \\ 0 & 0 \end{pmatrix}.$$

As the distance from γ_R to the critical point is ε, we have

$$\|P(\varepsilon)\| \sim \frac{1/2}{s(R)}.$$

Likewise, if $p(\lambda) = \lambda^d - 1$ we could take $R = 1 - \varepsilon^d$ and e.g. the truncated backward shift and perturb it slightly:

$$S(\varepsilon) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ . & . & . & . & . \\ . & . & . & . & . \\ & & & 0 & 1 \\ \varepsilon^d & 0 & . & . & 0 \end{pmatrix}.$$

Again, the eigenvalues are at distance ε from the origin and the projection to the direction of an eigenvector behaves like

$$\|P(\varepsilon)\| \sim \frac{1/d}{s(R)^{d-1}}.$$

In fact,

$$P(\varepsilon) = \frac{1}{d\varepsilon^{d-1}} \begin{pmatrix} \varepsilon^{d-1} & \varepsilon^{d-2} & . & . & 1 \\ \varepsilon^d & \varepsilon^{d-1} & . & . & \varepsilon \\ . & . & . & . & . \\ . & . & . & . & . \\ \varepsilon^{2d-2} & \varepsilon^{2d-3} & . & . & \varepsilon^{d-1} \end{pmatrix}.$$

In this case we would take the analytic function f to be identically 1 in the component which contains the point 1 and in the others we set it equal to zero. Then again $f(S(\varepsilon)) = P(\varepsilon)$.

2.3 Application to power boundedness

We can apply Theorem 2.3 with $R = 0$ but then A has to be an algebraic operator and all eigenvalues are nondefective. Requiring $p(A) = 0$ with p simple zeros says exactly that.

Consider now *power bounded* operators. Let $\mathbb{D}$ denote the open unit disc. Suppose that

$$V_p(A) \subset \overline{\mathbb{D}}. \tag{2.6}$$

If $p(\lambda) = (\lambda - 1)^2$ and

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then $p(A) = 0$, $V_p(A) = \{1\} \subset \overline{\mathbb{D}}$ but A is not power bounded as

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Observe that if p has multiple zeros, then $w = 0$ is a critical value. Thus we need to exclude this case.

Corollary 2.5. *Let p be monic with simple zeros and suppose $R \geq 0$ is such that γ_R contains no critical points and $\gamma_R \subset \overline{\mathbb{D}}$. If $A \in B(H)$ is such that $\|p(A)\| \leq R$, then A is power bounded and with the constant $C(R)$ provided by Theorem 1.1 we have for all $n \geq 1$*

$$\|A^n\| \leq C(R) \sum_{k=1}^{d} \|\delta_k(A)\|. \tag{2.7}$$

Proof. The claim follows from

$$\max_{z \in \gamma_R} |z^n| \leq 1.$$

Notice that here the fact that the constant is larger than 1 is not important but instead we need that the *same* constant works for all holomorphic f. ☐

Remark 2.6. If A is *algebraic* then we can take p to be the *minimal polynomial.* Recall that a polynomial is called minimal polynomial if it is monic, $p(A) = 0$ and the polynomial is of smallest possible degree. Then Corollary 2.5 and Remark 1.2 yield

$$\|A^n\| \leq \sum_{k=1}^{d} \|\delta_k(A)\|. \tag{2.8}$$

Observe that this can be obtained directly as follows. For every n there exists a polynomial q_n such that

$$z^n = \sum_{k=1}^{d} \lambda_k^n \delta_k(z) + q_n(z)p(z)$$

which implies (2.8), as $p(A) = 0$.

Example 2.7. Let $B(\varepsilon) = \frac{1}{\varepsilon} S(\varepsilon)$ with $S(\varepsilon)$ as in the Example 2.4. Then

$$B(\varepsilon)^d = I.$$

In particular, $B(\varepsilon)$ is power bounded: for $k \geq 0$ and $0 \leq m \leq d-1$

$$\|B(\varepsilon)^{kd+m}\| = \frac{1}{\varepsilon^m}. \tag{2.9}$$

The minimal polynomial,

$$p(\lambda) = \lambda^d - 1$$

is independent of ε and so are the basis polynomials $\delta_k(z)$. As $\varepsilon \to 0$ we have

$$\|\delta_k(B(\varepsilon))\| = \frac{1}{d\, \varepsilon^{d-1}}(1 + o(1))$$

and

$$\sum_{k=1}^{d} \|\delta_k(B(\varepsilon))\| = \frac{1}{\varepsilon^{d-1}}(1 + o(1))$$

so that the bound in (2.8) gives

$$\|B(\varepsilon)^n\| \leq \frac{1}{\varepsilon^{d-1}}(1 + o(1)),$$

comparing well with (2.9).

2.4 Other extensions

K. Okubo and T. Andô [8] have showed that if the numerical range is in the closed unit disc, then the disc is K-spectral with constant $K = 2$. Recall that the numerical range

$$W(A) = \{(Ax, x) \in \mathbb{C} \ : x \in H, \ \|x\| = 1\}$$

of $A \in B(H)$ is always convex, contains the spectrum and is included in the disc $|z| \leq \|A\|$. Denote by $w(A)$ the numerical radius of A:

$$w(A) = \sup_{z \in W(A)} |z|.$$

Then we can formulate the following modification of Theorem 2.3.

Corollary 2.8. *If the assumptions of Theorem 2.3 hold with the only exception that the condition $\|p(A)\| \leq R$ is relaxed to $w(p(A)) \leq R$, then the conclusion holds with constant K satisfying*

$$K \leq 2\, C(R) \sum_{k=1}^{d} \|\delta_k(A)\|.$$

More recently B. and F. Delyon [2] have generalized Okubo's and Andô's result from the disc to arbitrary compact convex sets.

Theorem 2.9. *(B. and F. Delyon [2])*
If Σ is a convex compact set such that $W(A) \subset \Sigma$ then there exists a constant K_Σ such that

$$\|R(A)\| \leq K_\Sigma \sup_{z \in \Sigma} |R(z)| \tag{2.10}$$

for all rational R holomorphic in Σ.

Remark 2.10. M. Crouzeix [3] has shown that $K_\Sigma \leq 11.08$. He conjectured that $K_\Sigma = 2$ will always do.

We can extend this result in the same way as that of Okubo and Andô. In fact, let $p \in \mathbb{P}_d$ be a monic polynomial with distinct zeros and $A \in B(H)$ be given. Put $B = p(A)$ and assume $W(B) \subset \Sigma$ with Σ convex and compact with a boundary that contains no critical points of p. Then we have

$$\|f_k(B)\| \leq K_\Sigma \sup_{w \in \Sigma} |f_k(w)| \tag{2.11}$$

Suppose we have an estimate of the form

$$\sup_{w \in \Sigma} |f_k(w)| \leq C_\Sigma \sup_{z \in p^{-1}(\Sigma)} |f(z)|. \tag{2.12}$$

Then we can combine these two inequalities as follows.

Theorem 2.11. *Suppose we are given a monic $p \in \mathbb{P}_d$ with distinct roots and a bounded operator $A \in B(H)$ in a Hilbert space H. Let Σ be a convex compact set such that $W(p(A)) \subset \Sigma$ and such that the boundary of Σ contains no critical values of p. Then for all f which are holomorphic in $p^{-1}(\Sigma)$ we have*

$$\|f(A)\| \leq C_\Sigma K_\Sigma \sum_{k=1}^{d} \|\delta_k(A)\| \sup_{z \in p^{-1}(\Sigma)} |f(z)|. \tag{2.13}$$

Proof. All we need is the existence of C_Σ in (2.12). We include this as a remark at the end of the proof of Theorem 1.1. $\square$

Remark 2.12. Clearly many of the existing results invite generalizations of this nature. To name still one, in [1] the authors have considered operators such that several discs are simultaneously spectral sets and shown that their intersection is then K-spectral. Again, one could pose the assumptions on $p(A)$ and then consider the boundary of the intersection of the related discs. If the boundary is free from critical values, the preimage of the intersection is again K-spectral with some K.

Remark 2.13. Recall that an operator $A \in B(H)$ is called *polynomially bounded* if for some K

$$\|P(A)\| \leq K \sup_{|z|\leq 1} |P(z)| \tag{2.14}$$

holds for all polynomials P. Such an operator is not necessarily similar to a contraction. To that the inequality has to hold not only for scalar polynomials but for polynomials with matrix valued coefficients. Such operators are called *completely polynomially bounded* and it is clear by the proof technique of using the multicentric representation that the versions with matrix valued functions extend in the very same way as the scalar ones.

3 Proof of Theorem 1.1

In this section we discuss the estimation problem: if

$$f(z) = \sum_{k=1}^{d} \delta_k(z) f_k(w) \quad where \quad w = p(z), \tag{3.1}$$

under what conditions we can bound f_k in terms of f. In particular, the discussion gives a proof for Theorem 1.1.

Proposition 3.1. *If f is holomorphic inside γ_{R_0}, then for $k = 1, \ldots, d$ the functions f_k are holomorphic for $|w| < R_0$.*

Proof. In [6] we discussed the multicentric representation by first writing the multicentric decomposition for the Cauchy kernel. This yields a separate kernel $K_k(z, w)$ for each center λ_k

$$K_k(z, w) = \frac{1}{z - \lambda_k} \; \frac{p(z)}{p(z) - w}.$$

It then follows that for $|w| < R < R_0$ we have

$$f_k(w) = \frac{1}{2\pi i} \int_{\gamma_R} K_k(z, w) f(z) dz. \tag{3.2}$$

$\square$

The integral representation (3.2) leads to bounds of the form

$$\sup_{|w|\leq r} |f_k(w)| \leq C(R,r) \sup_{|p(z)|\leq R} |f(z)|.$$

with $r < R$ but here we are interested in having $r = R$. In order to prove such a bound we consider a different type of explicit representation for f_k in terms of f.

Assume first that w is a noncritical value of p and denote the d different roots of $p - w$ by $\zeta_j = \zeta_j(w)$

$$p(\zeta_j) - w = 0$$

with numbering such that $\zeta_j(w) \to \lambda_j$ as $w \to 0$. Next, let $\varepsilon_j \in \mathbb{P}_{d-1}$ be the polynomials such that they equal 1 at ζ_j and vanish at the other roots ζ_l

$$\varepsilon_j(\zeta_l) = \delta_{jl}. \tag{3.3}$$

The roots ζ_l are analytic functions of w away from the critical values, and so are the coefficients of ε_j, too.

Proposition 3.2. *If w is noncritical, then*

$$f_k(w) = \sum_{j=1}^{d} \varepsilon_j(\lambda_k) f(\zeta_j(w)). \tag{3.4}$$

Proof. For fixed w we introduce two polynomials, P and Q as follows

$$P(\zeta) = \sum_{k=1}^{d} \delta_k(\zeta) f_k(w)$$

while

$$Q(\zeta) = \sum_{j=1}^{d} \varepsilon_j(\zeta) f(\zeta_j).$$

By (3.1)

$$f(\zeta_j) = \sum_{k=1}^{d} \delta_k(\zeta_j) f_k(w) = P(\zeta_j)$$

and therefore

$$Q(\zeta) = \sum_{j=1}^{d} \varepsilon_j(\zeta) P(\zeta_j).$$

But then P and Q are the same polynomial. Substituting $\zeta = \lambda_k$ we obtain

$$Q(\lambda_k) = \sum_{j=1}^{d} \varepsilon_j(\lambda_k) f(\zeta_j) = P(\lambda_k) = f_k(w).$$

□

Assume now that f is holomorphic inside and on γ_R and let w_0 be a critical value of p such that $|w_0| < R$. For simplicity, let us assume that z_0 is the only critical point such that

$$p(z_0) = w_0.$$

The modifications needed with several such critical points are obvious and left to the reader. Assume that the roots $\zeta_j(w)$ are numbered such that for $j = 1, \ldots, m$ we have $\zeta_j(w) \to z_0$ as $w \to w_0$ while the other roots stay within a positive distance from the critical point. Then we may denote

$$\zeta_j(w) = z_0 + (\zeta_1(w) - z_0)e^{2\pi i \frac{j-1}{m}}(1 + o(1)). \tag{3.5}$$

By Proposition 3.1. we know that w_0 must be a removable singularity of

$$w \mapsto \sum_{j=1}^{m} \varepsilon_j(\lambda_k) f(\zeta_j(w)). \tag{3.6}$$

In order to compute the limit, we use the explicit representation of the polynomials:

Lemma 3.3. *At noncritical w we have*

$$\varepsilon_j(\zeta) = \frac{p(\zeta) - w}{p'(\zeta_j(w))(\zeta - \zeta_j(w))}.$$

In particular, as $p(\lambda_k) = 0$,

$$\varepsilon_j(\lambda_k) = \frac{w}{p'(\zeta_j(w))(\zeta_j(w) - \lambda_k)}.$$

Proof. This is just the usual formula for the basis polynomials in Lagrange interpolation, applied to the nodes $\zeta_l(w)$. $\square$

Hence the individual polynomials ε_j are not bounded as $\zeta_j(w) \to z_0$. However, we have with some $c \neq 0$

$$p'(\zeta_j(w)) = (\zeta_j(w) - z_0)^{m-1}c + O((\zeta_j(w) - z_0)^m)$$
$$w = w_0 + O((\zeta_j(w) - z_0)^m)$$
$$\zeta_j(w) - \lambda_k = (z_0 - \lambda_k) + (\zeta_j(w) - z_0),$$

and after expanding $f(\zeta_j(w))$ into power series around z_0 and substituting all expansions into (3.6) we see that the key terms are of the form

$$\frac{const}{(\zeta_1(w) - z_0)^{m-1-n}} \sum_{j=1}^{m} e^{-2\pi i \frac{j-1}{m}(m-1-n)}$$

with $n \geq 0$. As long as $n < m - 1$ all sums vanish and hence (3.6) stays bounded as $w \to w_0$.

Suppose now that γ_R does not contain any critical points. Then for $|w| < R$

$$|f_k(w)| \leq C_k(R) \sup_{z \in \gamma_R} |f(z)|$$

with

$$C_k(R) = \sup_{|w|=R} \sum_{j=1}^{d} |\varepsilon_j(\lambda_k)|. \tag{3.7}$$

What remains is to estimate $C(R)$ from (3.7).

From Lemma 3.3 we see that near w_0

$$|\varepsilon_j(\lambda_k)| \sim \frac{|w_0|}{|z_0 - \lambda_k|} \frac{1}{s(R)^{m-1}}$$

where we denote by $s(R)$ the distance from γ_R to the nearest critical point. Further, as $R \to 0$ we have $\varepsilon_j(\lambda_k) \to \delta_{jk}$, and so $C_k(R) \to 1$. When $R \to \infty$, we have

$$\varepsilon_j(\lambda_k) = \frac{1 + o(1)}{d}$$

and again $C_k(R) \to 1$. Hence, there exists C such that for all $R \geq 0$

$$C_k(R) \leq 1 + \frac{C}{s(R)^{d-1}}.$$

Naturally, generically the critical points are simple and in such a case one can bound $C_k(R)$ by $1/s(R)$.

Example 3.4. Let $p(z) = z^2 - 1$ and consider the lemniscate with $R = 1$ so that γ_1 passes through the origin which is a critical point of p. We show that there exists no finite $C(1)$ such that

$$\sup_{|z^2-1|\leq 1} |f_k(z^2 - 1)| \leq C(1) \sup_{|z^2-1|\leq 1} |f(z)|. \tag{3.8}$$

would hold for all f holomorphic inside and on γ_1. One checks easily that if we set $\lambda_k = (-1)^{k+1}$, then

$$f_k(z^2 - 1) = \frac{1}{2}(f(z) + f(-z)) + (-1)^{k+1} \frac{1}{2z}(f(z) - f(-z)).$$

Consider the one-parameter family of functions with $\varepsilon > 0$:

$$f(z; \varepsilon) = \frac{\varepsilon}{z - i\varepsilon}.$$

Then we have, as $\varepsilon \to 0$,

$$\sup_{|z^2-1|\leq 1} |f_k(z^2 - 1; \varepsilon)| \sim \frac{const}{\varepsilon},$$

while

$$\sup_{|z^2-1|\leq 1} |f(z; \varepsilon)| = \mathcal{O}(1).$$

Example 3.5. We can use the previous example to also demonstrate what happens in estimating $f(A)$ if the lemniscate contains a critical point. Let $f(\ ; \varepsilon)$ and p be as in the previous example and take

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.9}$$

so that $p(A) = -I$. Then in particular $\|p(A)\| = 1$ and the eigenvalue of A is at the critical point. Again, as $\varepsilon \to 0$ we have

$$\sup_{|z^2-1|\leq 1} |f(z; \varepsilon)| = \mathcal{O}(1),$$

while

$$\|f(A;\varepsilon)\| \sim \frac{const}{\varepsilon}.$$

So, there exists no constant K such that for all $\varepsilon > 0$

$$\|f(A;\varepsilon)\| \leq K \sup_{|p(z)|\leq 1} |f(z;\varepsilon)|.$$

In particular, the requirement, that γ_R contains no critical points, cannot be omitted in Theorem 2.3.

Remark 3.6. For the proof of Theorem 2.11 we need to establish the bound (2.12). However, all we need is to have a bound for

$$w \mapsto \sum_{j=1}^{d} |\varepsilon_j(\lambda_k)|$$

along the boundary of Σ. But as the boundary does not pass through any critical value, this function is continuous, and since the boundary is compact, the function is bounded.

Acknowledgement
The author wants to thank Michel Crouzeix for helpful comments on a draft version.

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ISBN: 978-952-60-4162-9
ISSN-L: 1799-4896
ISSN: 1799-490X (pdf)
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